FINITE GENERATION OF THE COHOMOLOGY OF QUOTIENTS OF PBW ALGEBRAS

A Dissertation

by

PIYUSH RAVINDRA SHROFF

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2012

Major Subject: Mathematics

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Chair of Committee,	Sarah Witherspoon
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ABSTRACT

Finite Generation of Cohomology of Quotients of
PBW Algebras. (August 2012)

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In this dissertation we prove finite generation of the cohomology of quotients of a PBW algebra denoted by A by relating it to the cohomology of quotients of a quantum symmetric algebra denoted by S which is isomorphic to the associated graded algebra of A. The proof uses a spectral sequence argument and a finite generation lemma adapted from Friedlander and Suslin.

To Late Ratilal Chasmawala, Ravindra Shroff, Ila Shroff,
Janhavi Chaturvedi and Atul Chaturvedi

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CHAPTER I

INTRODUCTION

Cohomology of algebras contains lots of information as a whole. Especially in commutative algebra, many properties of algebras in which people are interested have homological interpretations making it easier to organize information. Any knowledge of homology or cohomology of an algebra thus potentially aides us in providing information about the algebra and its modules. The property of being finitely generated is very important because it is much easier to understand a finitely generated algebra. Knowing that cohomology is finitely generated leads us to know nicer things about the algebra. A finitely generated commutative algebra is useful for geometric study via algebraic geometry.

The cohomology ring of a finite group is finitely generated, as proven by Evens [7], Golod [9] and Venkov [19]. The door to use geometric methods in the study of cohomology and modular representations of finite groups was opened due to this fundamental result. The cohomology ring of any finite group scheme (equivalently, finite dimensional cocommutative Hopf algebra) over a field of positive characteristic is finitely generated, as proven by Friedlander and Suslin [8] which is a generalization of the result of Venkov and Evens. In [11], Ginzburg and Kumar proved that cohomology of quantum groups at roots of unity is finitely generated. In [6], Etingof and Ostrik conjectured finite generation of cohomology in the context of finite tensor categories. The task of proving this conjecture was done by Mastnak, Pevtsova, Schauenburg and Witherspoon [15] for some classes of noncocommutative Hopf algebras over a field of characteristic 0 .

This dissertation follows the style of *Jounal of Algebra.*

In [15], Mastnak, Pevtsova, Schauenburg and Witherspoon considered arbitrary data with the corresponding Yetter-Drinfeld module V and the Nichols algebra R. A finite filtration on R is used to define a spectral sequence to which they apply a finite generation lemma adapted from [8]. In order to do so, they find some permanent cycles which leads them to define 2-cocycles on R. These 2-cocycles were previously studied in [16]. Finally, they identify the permanent cycles belonging to the degree 2 cohomology of the associated graded algebra of R with ξ_i in the cohomology of S (where S is a quantum symmetric algebra subject to the relation $x_i^{N_i} = 0$ for all i) constructed in Section 4 of [15].

In this dissertation, we generalize the work done by Mastnak, Pevtsova, Schauenburg and Witherspoon [15] by choosing our parameters that are not necessarily roots of unity and we allow non-nilpotent generators. Also we deal with PBW algebras in general, whereas in [15] authors looked at those that arise from subalgebras of pointed Hopf algebras. Let k be a field, usually assumed to be algebraically closed and of characteristic 0. Let B be a PBW algebra over k generated by $x_1, \cdots, x_\theta, \cdots, x_n$ and $A = B/(x_1^{N_1}, \cdots, x_\theta^{N_\theta})$ where for each i, $1 \le i \le \theta$, N_i is an integer greater than 1 and $x_i^{N_i}$ is in the braided center. Our proof of finite generation of cohomology of the algebra A, is a two step procedure. First for these algebras, we compute cohomology explicitly via a free S-resolution where S is a quotient of a quantum symmetric algebra by the ideal generated by $x_1^{N_1}, \cdots, x_\theta^{N_\theta}$ where $1 \le \theta \le n$. Second, our algebra A has a filtration [4, Theorem 4.6.5] for which the associated graded algebra (GrA) is S.

This work can be applied to Frobenius-Lusztig kernels studied by Drupieski [5] and pointed Hopf algebras studied by Helbig [12].

Notation: $H^r(A,k) = Ext^r_A(k,k)$ and $H^*(A,k) = \bigoplus_{r\ge 0} H^r(A,k)$.

Main Theorem: The cohomology algebra $H^*(A,k)$ is finitely generated over the subalgebra generated by all ξ_i.

We use the techniques of Mastnak, Pevtsova, Schauenberg and Witherspoon [15] to yield results in this general setting. However, some differences do arise, namely we cannot apply [8, Lemma 1.6] as it is since our parameters are not necessarily roots of unity, we do not get commutativity.

We now describe the contents of this dissertation. In Chapter 2 PBW algebras with various other algebras are introduced. In addition, we introduce a result from Evens [7] and a non-commutative version of a finite generation lemma adapted from Friedlander and Suslin [8]. In Chapter 3 we prove that cohomology of quotients of a quantum symmetric algebra S is finitely generated. Chapter 4 introduces a 2-cocycle on the algebra A. In Chapter 5 we prove that cohomology of the algebra A is finitely generated.

CHAPTER II

DEFINITIONS AND PRELIMINARY RESULTS

A. PBW Algebras

In this section we recall some basic definitions to define a PBW algebra.

Definition II.1. *An admissible ordering on $\mathbb{N}^n$ is a total ordering $<$ such that*

1) if $\alpha < \beta$ and $\gamma \in \mathbb{N}^n$ then $\alpha + \gamma < \beta + \gamma$

2) $<$ is a well ordering.

Some examples of ordering on n-tuples include:

Example II.2. *Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}^n$. The lexicographic order $<_{lex}$ on $\mathbb{N}^n$ is defined by letting $\beta <_{lex} \alpha$ if the first non zero entry of $\alpha - \beta \in \mathbb{Z}^n$ is positive. We will write $x^\beta <_{lex} x^\alpha$ if $\beta <_{lex} \alpha$.*

Illustrated below are some examples of lexicographic order:

a. $(3, 2, 4) <_{lex} (3, 3, 0)$ because $\alpha - \beta = (0, 1, -4)$.

b. $(0, 3, 4) <_{lex} (1, 2, 0)$ because $\alpha - \beta = (1, -1, -4)$.

c. The variables $x_1, \cdots, x_n$ are ordered in the usual way by the lex ordering:

$$(0, 0, \cdots, 1) <_{lex} \cdots <_{lex} (0, 1, \cdots, 0) <_{lex} (1, 0, \cdots, 0).$$

So $x_n <_{lex} \cdots <_{lex} x_2 <_{lex} x_1$.

This is just one example of ordering on n-tuples. For more examples refer to [4].

In light of this definition and example we define a PBW algebra.

Poincaré Birkhoff Witt Algebra: A PBW algebra R, over a field k, is a k-algebra together with elements $x_1, \cdots, x_n \in R$ and an admissible order on $\mathbb{N}^n$ for which there are scalars $q_{ij} \in k^*$ such that

1) $\{x_1^{\alpha_1}\cdots x_n^{\alpha_n} \mid (\alpha_1,\cdots,\alpha_n)\in\mathbb{N}^n\}$ is a basis of R as a k-vector space. We call this basis the PBW basis.

2) $x_ix_j = q_{ij}x_jx_i + p_{ij}$ for $p_{ij}\in R$ with $exp(p_{ij}) < \varepsilon_i+\varepsilon_j$ $(1\le i<j\le n)$ where $\varepsilon_i = (0,\cdots,0,1_i,0,\cdots,0)\in\mathbb{N}^n$.

Notation: By the basis condition of the definition, every $f\in R$ may be written uniquely as $f=\sum_{\alpha\in\mathbb{N}^n} c_\alpha x^\alpha$ (notation $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$) and $exp(f) = max\{\alpha\in\mathbb{N}^n \mid c_\alpha\neq 0\}$.

Let us now give some examples of PBW algebras.

Example II.3. *1) The polynomial ring $R=k[x_1,x_2,\cdots,x_n]$ is a PBW algebra.*

2) Let A be the Weyl algebra over a field k that is,

$$A = k\langle x_1,x_2,\cdots,x_n,y_1,\cdots,y_n \mid y_jx_i = x_iy_j-\delta_{ij},\ x_jx_i = x_ix_j,\ y_jy_i=y_iy_j\rangle$$

where δ_{ij} is the Kronecker delta. Then A is a PBW algebra.

3) There are some quantum groups which are PBW algebras. For example:

a. The quantum plane $k_q[x,y] = k\langle x,y\mid yx=qxy\rangle$

b. $U_q(sl_3)^+ := k\langle x_1,x_2,x_3 \mid x_1x_2 = qx_2x_1,\ x_2x_3 = qx_3x_2,\ x_1x_3 = q^{-1}x_3x_1+x_2\rangle$

The ω-filtration of a PBW algebra:

Let $\omega = (\omega_1,\cdots,\omega_n)\in\mathbb{N}^n$. For $0\neq f$ belonging to a PBW algebra R we define its ω-degree as

$$\deg_\omega(f) = \max\{|\alpha|_\omega \mid \alpha\in\mathcal{W}\}$$

where $|\alpha|_\omega = \alpha_1\omega_1+\cdots+\alpha_n\omega_n$, $f=\sum_{\alpha\in\mathbb{N}^n} c_\alpha x^\alpha$ and $\mathcal{W} = \{\alpha\in\mathbb{N}^n \mid c_\alpha\neq 0\}$. With these notations we define the ω-filtration of a PBW algebra as

$$F_s^\omega R = \{f\in R \mid |\alpha|_\omega \le s \text{ for all } \alpha\in\mathcal{W}\}$$

where s is any nonnegative integer.

In Chapters 3, 4 and 5 we will use the following several different types of algebras:

Quantum Symmetric Algebra: Let k be a field. Let n be a positive integer and for each pair i, j of elements in $\{1, \cdots, n\}$, let q_{ij} be a nonzero scalar such that $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j. Denote by $\mathbf{q}$ the corresponding tuple of scalars, $\mathbf{q} := (q_{ij})_{1 \leq i < j \leq n}$. Let V be a vector space with basis $x_1, \cdots, x_n$, and let

$$S_{\mathbf{q}}(V) := k\langle x_1, \ldots, x_n \mid x_i x_j = q_{ij} x_j x_i \text{ for all } 1 \leq i < j \leq n \rangle,$$

the *quantum symmetric algebra* determined by $\mathbf{q}$.

This is another example of PBW algebra.

Filtered Algebra: An algebra A over a field k is said to be a filtered k-algebra, if it is endowed with an ascending chain $FA = \{F_n A; n \geq 0\}$ of subvector-spaces, the "filtration" of A, satisfying for all $n, m \geq 0$

(1)$1 \in F_0 A$;

(2)$F_n A \subseteq F_{n+1} A$;

(3)$(F_n A)(F_m A) \subseteq F_{n+m} A$;

(4)$A = \bigcup_{n \geq 0} F_n A$.

The definition of filtered algebra with descending filtration can be obtained by naturally converting our definition of a filtered algebra. This is an algebra A endowed with a descending chain $FA = \{F^n A; n \geq 0\}$ of subvector-spaces such that $A = F^0 A$, $F^{n+1} A \subseteq F^n A$ and $(F^n A)(F^m A) \subseteq F^{n+m} A$ for all $n, m \geq 0$ [14].

Given a ring R, a decreasing filtration $F^n R$ for $n \in \mathbb{N}$ is called compatible with the ring structure on R if $F^m R \cdot F^n R \subset F^{m+n} R$, for all $m, n \in \mathbb{N}$. The ring R with this filtration is then called a filtered ring [3].

Associated Graded Algebra: If A is a filtered algebra then the associated graded algebra is defined as $Gr^F A = \bigoplus_{n \geq 0} (Gr^F A)_n$ where $(Gr^F A)_0 = F_0 A$ and for all $n > 0, (Gr^F A)_n = F_n A / F_{n-1} A$. The multiplication is defined by $(x + F_{n-1} A)(y +$

$F_{m-1}A) = xy + F_{n+m-1}A$ where $x \in F_nA$ and $y \in F_mA$.

Augmented Algebra: An augmented algebra over a field k is a k-algebra A together with an algebra homomorphism $\varepsilon : A \to k$.

Example II.4. $A = k[x_1, x_2, \cdots, x_n]$, $\varepsilon(x_i) = 0$ *for all* i *and* $\varepsilon(a) = a$ *for all* $a \in k$.

B. Noetherian modules

Let $R = F^0R \supseteq F^1R \supseteq \cdots \supseteq F^sR \supseteq \cdots$ be a graded filtered ring. Note that by definition, the grading on R is compatible with the ring structure in the usual way that is $R = \bigoplus_{n\in\mathbb{N}} R^n$, and $R^nR^m \subset R^{n+m}$. Then we may form the doubly graded ring

$$E_0(R) = \sum_i F^iR/F^{i+1}R.$$

Similarly we may form the doubly graded module $E_0(N)$ over $E_0(R)$ if N is a singly graded filtered module over R (with the module structure consistent with the ring structure in the usual way that is $N = \bigoplus_{i\in\mathbb{N}} N^i$, and $R^iN^j \subset N^{i+j}$).

For the current purposes it is sufficient to consider filtrations such that $F^iR^n = 0$ for i sufficiently large where n denotes the grading on R. Similarly, $F^iN^j = 0$ for i sufficiently large.

Since N is a graded module, $N = \bigoplus_i N^i$. The non-zero elements of N^i are called homogeneous elements of degree i.

We recall the following proposition of Evens [7]. We include a proof for completeness but before that we define a couple of terms that will help us to understand the proof.

Definition II.5. *A submodule* $S \subset N$ *is said to be homogeneous if it is generated by homogeneous elements (i.e. the elements from homogeneous summands* N^i*).*

Definition II.6. *An R-submodule N of the graded R-module M is called a graded R-submodule of M if we have $N = \bigoplus_s (N \cap M^s)$.*

Definition II.7. *If $\{F^s M\}$ is a filtration of the R-module M, and N is a submodule of M, then we have a filtration induced on N, given by $F^s N = N \cap F^s M$.*

Proposition II.8. *Let R be a graded filtered ring i.e.*

$$R = F^0 R \supseteq F^1 R \supseteq \cdots \supseteq F^s R \supseteq \cdots$$

and N a graded filtered R module i.e. suppose

$$N = F^0 N \supseteq F^1 N \supseteq \cdots \supseteq F^s N \supseteq \cdots$$

over R. If $E_0(N)$ is (left) Noetherian over $E_0(R)$, then N is Noetherian over R.

Proof. We want to show that the ascending chain condition (ACC) holds for R-submodules of N, that is, to show that any chain of R-submodules of N : $N_0 \subset N_1 \subset \cdots N_s \subset N_{s+1} \cdots$ stops. First, reduce to the case of homogeneous submodules as follows. If L is a submodule of N, let $\tilde{L} = \oplus L^n$ where L^n consists of 0 and all the elements in N^n which are highest degree components of elements in L. Then by definition II.5 we can see that $\tilde{L}$ is a homogeneous submodule of N. So if $L' \subseteq L''$ and $\tilde{L'} = \tilde{L''}$, then $L' = L''$ (since highest degree components of elements in L'' are contained in L'). That is enough to establish the reduction. For the homogeneous submodules, the proof reduces similarly to showing that if $L' \subseteq L''$ and $E_0(L') = E_0(L'')$, then $L' = L''$. This will follow easily by showing inductively that $L'/F^p L' = L''/F^p L''$ for each p, and then noting that, in any homogeneous component, $F^p L'^n = 0$ for sufficiently large p.

By the definition of E_0, assuming $E_0(L') = E_0(L'')$, we have $F^0 L'/F^1 L' \cong F^0 L''/F^1 L''$, that is, $L'/F^1 L' \cong L''/F^1 L''$. In fact, $F^{p-1} L'/F^p L' \cong F^{p-1} L''/F^p L''$

for each p.

Let $\alpha : L' \hookrightarrow L''$ be the inclusion map. Define ψ_1 in the following diagram to be the map induced from α. Define ϕ_1, ϕ_2 to be quotient maps respectively. Let $\psi = \phi^{-1} \circ \phi_2$ where ϕ is also induced from an inclusion map and is an isomorphism since $(L'/F^2L')/(F^1L'/F^2L') \cong L'/F^1L'$ and $(L''/F^2L'')/(F^1L''/F^2L'') \cong L''/F^1L''$. Consider the following commuting diagram

$$\begin{tikzcd} L'/F^2L' \arrow[r, "\psi_1"] \arrow[d, "\phi_1"'] & L''/F^2L'' \arrow[d, "\phi_2"] \arrow[dl, "\psi"'] \\ L'/F^2L'/F^1L'/F^2L' \arrow[r, "\cong", "\phi"'] & L''/F^2L''/F^1L''/F^2L'' \end{tikzcd}$$

We want show that ψ_1 is an isomorphism.

First we show that ψ_1 is surjective.

Let $a+F^2L'' \in L''/F^2L''$. Then $\psi(a+F^2L'') = b+F^1L'/F^2L'$, where $b = b_1+F^2L'$ for some $b_1 \in L'$. So $\phi_1(b_1 + F^2L') = b + F^1L'/F^2L'$. Since α is a inclusion map we have $\alpha(b_1) = b_1 \in L''$ and therefore, $\psi_1(b_1 + F^2L') = b_1 + F^2L''$.

Then, $\psi(b_1 + F^2L'') = b + F^1L'/F^2L'$. Thus

$$\psi(a + F^2L'') - \psi(b_1 + F^2L'') = 0$$

$$\text{i.e. } \psi((a - b_1) + F^2L'') = 0$$

$$\text{i.e. } \psi((a - b_1) + F^2L'') \subset F^1L'/F^2L'$$

$$\text{i.e. } (a - b_1) \in F^1L'$$

Let $a - b_1 =: c \in F^1L'$. Therefore, $(a - b_1) + F^2L' = c + F^2L' \in F^1L'/F^2L' \cong F^1L''/F^2L''$. So $a + F^2L'' = b_1 + c + F^2L'' = \alpha(b_1 + c) + F^2L''$. Thus $\alpha(b_1 + c) = a$. Therefore, for any $a + F^2L'' \in L''/F^2L''$ we have $b_1 + c + F^2L' \in L'/F^2L'$ such that $\psi_1(b_1 + c + F^2L') = a + F^2L''$. Therefore, ψ_1 is surjective.

Next we show that Ker $\psi_1 = 0$ i.e. ψ_1 is injective. This follows from the fact that $F^2L' \subset F^2L''$ and $F^2L'' \cap L' = F^2L'$.

Hence $L'/F^2L' \cong L''/F^2L''$. So by induction we have $L'/F^pL' = L''/F^pL''$ for all p. For p sufficiently large, $F^pL'^n = 0$; hence, we get $L' = L''$.

□

Now we state some standard results from [10] about noncommutative Noetherian rings.

Theorem II.9 (10, Corollary 1.3)**.** *Any finite direct sum of Noetherian modules is Noetherian.*

Theorem II.10 (10, Corollary 1.4)**.** *If R is a right Noetherian ring, all finitely generated right R-modules are Noetherian.*

Theorem II.11 (10, Corollary 1.5)**.** *Let $\mathcal{S}$ be subring of a ring R. If $\mathcal{S}$ is right Noetherian and R is finitely generated as a right $\mathcal{S}$-module, then R is right Noetherian.*

A finite generation lemma. In Chapter 5, we will need the following general lemma which is a non-commutative version of [8, Lemma 1.6] and [15, Lemma 2.5]. Recall that an element $x \in E_r^{p,q}$ is called a *permanent cycle* if $d_i(x) = 0$ for all $i \geq r$. More precisely, if $i > r$, d_i is applied to the image of x in E_i.

Lemma II.12. *a) Let $E_1^{p,q} \Rightarrow E_\infty^{p+q}$ be a multiplicative spectral sequence of bigraded k-algebras concentrated in the half plane $p+q \geq 0$ and let $C^{*,*}$ be a bigraded k-algebra. For each fixed q, assume that $C^{p,q} = 0$ for p sufficiently large. Assume that there exists a bigraded map of algebras $\phi : C^{*,*} \to E_1^{*,*}$ such that*

1) ϕ makes $E_1^{*,*}$ into a left Noetherian $C^{*,*}$-module, and

2) the image of $C^{*,*}$ in $E_1^{*,*}$ consists of permanent cycles.

Then E_∞^* is a left Noetherian module over Tot($C^{*,*}$).

b) Let $\widetilde{E}_1^{p,q} \Rightarrow \widetilde{E}_\infty^{p+q}$ be a spectral sequence that is a bigraded module over the spectral sequence $E^{*,*}$. Assume that $\widetilde{E}_1^{*,*}$ is a left Noetherian module over $C^{*,*}$ where $C^{*,*}$ acts on $\widetilde{E}_1^{*,*}$ via the map ϕ. Then $\widetilde{E}_\infty^{*}$ is a finitely generated E_∞^{*}-module.

Proof. Let $\Lambda_r^{*,*} \subset E_r^{*,*}$ be the bigraded subalgebra of permanent cycles in $E_r^{*,*}$. We claim first that $d_r(E_r^{*,*}) \subset \Lambda_r^{*,*}$. In order to see this note that $d_r(E_r^{*,*}) = \text{im}(d_r)$. Therefore, $d_r(E_r^{*,*}) \subset \text{Ker } d_{r+1}$. Hence, $d_{r+1}(d_r(E_r^{*,*})) = 0$. Similarly, $d_{r+2}(d_r(E_r^{*,*})) = 0$ and so on. Thus, we have $d_i(d_r(E_r^{*,*})) = 0$ for all $i \geq r$. Hence, $d_r(E_r^{*,*}) \subset \Lambda_r^{*,*}$.

Next we claim that for all $\lambda \in \Lambda_r^{*,*}$ and $\mu \in E_r^{*,*}$, $\lambda \cdot d_r(\mu) \in d_r(E_r^{*,*})$ that is, $d_r(E_r^{*,*})$ is a left ideal of $\Lambda_r^{*,*}$. Consider

$$\begin{aligned} d_r(\lambda \cdot \mu) &= d_r(\lambda)\mu + (-1)^{p+q}\lambda \cdot d_r(\mu) \quad \text{where } \lambda \in \Lambda^{p,q} \\ &= 0 + (-1)^{p+q}\lambda \cdot d_r(\mu) \end{aligned}$$

So $\lambda \cdot d_r(\mu) \in d_r(E_r^{*,*})$. Thus $d_r(E_r^{*,*})$ is a left ideal of $\Lambda_r^{*,*}$.

Now the image of $C^{*,*}$ is contained in each page of the spectral sequence and by assumption it consists of permanent cycles. Hence, we can similarly conclude as above that $d_r(E_r^{*,*})$ is a $C^{*,*}$-submodule.

A similar computation as above shows that $\Lambda_1^{*,*}$ is a $C^{*,*}$-submodule of $E_1^{*,*}$. To see this let $a \in C^{p,q}$; therefore, $\phi(a) \in E_1^{*,*}$ and $\lambda_1 \in \Lambda_1^{*,*}$. Consider,

$$\begin{aligned} d_i(\phi(a)\lambda_1) &= d_i(\phi(a))\lambda_1 + (-1)^{p+q}\phi(a)d_i(\lambda_1) \quad \text{where } i \geq 1 \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

So $\phi(a)\lambda_1 \in \Lambda_1^{*,*}$. Thus $\Lambda_1^{*,*}$ is an $C^{*,*}$-submodule.

By induction, $\Lambda_{r+1}^{*,*} = \Lambda_r^{*,*}/d_r(E_r^{*,*})$ is an $C^{*,*}$-module for any $r \geq 1$ because

$d_r(E_r^{*,*}) \subset \Lambda_r^{*,*}$ and by the induction hypothesis $\Lambda_r^{*,*}$ is a $C^{*,*}$-module. Therefore, $\Lambda_r^{*,*}/d_r(E_r^{*,*})$ is a $C^{*,*}$-module that is, $\Lambda_{r+1}^{*,*}$ is a $C^{*,*}$-module.

We get a sequence of surjective maps of $C^{*,*}$-modules:

$$\Lambda_1^{*,*} \twoheadrightarrow \Lambda_2^{*,*} \twoheadrightarrow \cdots \twoheadrightarrow \Lambda_r^{*,*} \twoheadrightarrow \Lambda_{r+1}^{*,*} \twoheadrightarrow \cdots \tag{II.1}$$

Since $\Lambda_1^{*,*}$ is a $C^{*,*}$-submodule of $E_1^{*,*}$, it is Noetherian as a $C^{*,*}$-module. Therefore, the kernels of the maps $\Lambda_1^{*,*} \twoheadrightarrow \Lambda_r^{*,*}$ are Noetherian for all $r \geq 1$. These kernels form an increasing chain of submodules of $\Lambda_1^{*,*}$; hence, by the Noetherian property, they stabilize after finitely many steps; that is, $\Lambda_r^{*,*} = \Lambda_{r+1}^{*,*} = \cdots$ for some r. We conclude that $\Lambda_r^{*,*} = E_\infty^{*,*}$. Therefore $E_\infty^{*,*}$ is a Noetherian $C^{*,*}$-module. Also, both $E_\infty^{*,*}$ and $C^{*,*}$ are filtered algebras and the filtration for each n is given by:

$$E_\infty^n = \bigoplus_{p+q=n} E_\infty^{p,q} \supseteq \bigoplus_{\substack{p+q=n \\ p\geq 1}} E_\infty^{p,q} \supseteq \bigoplus_{\substack{p+q=n \\ p\geq 2}} E_\infty^{p,q} \supseteq \cdots$$

and $E_\infty^{*,*}$ is the associated graded algebra. Similarly, for each n:

$$C^n = \bigoplus_{p+q=n} C^{p,q} \supseteq \bigoplus_{\substack{p+q=n \\ p\geq 1}} C^{p,q} \bigoplus_{\substack{p+q=n \\ p\geq 2}} C^{p,q} \supseteq \cdots$$

and $C^{*,*}$ is the associated graded algebra.

For p sufficiently large, $C^{p,q} = 0$. Hence, by proposition II.8, E_∞^* is a Noetherian module over $\mathrm{Tot}(C^{*,*})$.

(b) Similarly, we can show that $\widetilde{E}_\infty^{*,*}$ is Noetherian over $C^{*,*}$. Again, by applying proposition II.8, we can conclude that $\widetilde{E}_\infty^*$ is Noetherian and hence finitely generated over $\mathrm{Tot}(C^{*,*})$. Therefore, by part (a) $\widetilde{E}_\infty^*$ is a Noetherian module over E_∞^*. Hence, $\widetilde{E}_\infty^*$ is finitely generated over E_∞^*.

□

CHAPTER III

COHOMOLOGY OF QUOTIENTS OF QUANTUM SYMMETRIC ALGEBRAS

For this chapter we will use the same terminology as used by Mastnak, Pevtsova, Schauenburg and Witherspoon in Section 4 of [15]. We will make some modifications to their method to accommodate non-nilpotent generators. This will enable us to generalize their method.

Let $n,\ \theta$ with $\theta \leq n$ be positive integers, and for each $i,\ 1 \leq i \leq \theta$, let $1 < N_i \in \mathbb{Z}$. Let $q_{ij} \in k^*$ for $1 \leq i < j \leq n$ with $q_{ji} = q_{ij}^{-1}$ for $i < j$ and $q_{ii} = 1$. Let

$$S = k\langle x_1, \cdots, x_\theta, \cdots, x_n \mid x_i x_j = q_{ij} x_j x_i \text{ for all } i < j \text{ and } x_i^{N_i} = 0 \text{ for } 1 \leq i \leq \theta\rangle. \tag{III.1}$$

We will compute $H^*(S, k) = Ext_S^*(k, k)$ where k is an S-module where x acts as multiplication by zero. By using the twisted tensor product formula of Bergh and Oppermann [2, Thms. 3.7 and 5.3] the structure of this ring may be determined. To obtain information at the chain level, we need an explicit free S-resolution of k. This resolution is originally adapted from [1] and it is a braided tensor product of the periodic resolutions

$$\cdots \xrightarrow{x_i^{N_i-1}\cdot} k[x_i]/(x_i^{N_i}) \xrightarrow{x_i\cdot} k[x_i]/(x_i^{N_i}) \xrightarrow{x_i^{N_i-1}\cdot} k[x_i]/(x_i^{N_i}) \xrightarrow{x_i\cdot} k[x_i]/(x_i^{N_i}) \xrightarrow{\varepsilon} k \to 0,$$

one for each $i,\ 1 \leq i \leq \theta$ and

$$0 \to k[x_i] \xrightarrow{x_i\cdot} k[x_i] \xrightarrow{\varepsilon} k \to 0,$$

one for each $i,\ \theta + 1 \leq i \leq n$.

Explicitly, we define a complex $K_\bullet$ of free S-modules as follows. For each n-tuple $(a_1, \cdots, a_n)$ of non-negative integers with $a_i = 0$ or 1 for each i, $\theta + 1 \leq i \leq n$, let $\Phi(a_1, \cdots, a_n)$ be a free generator in degree $a_1 + \cdots + a_n$. Thus

$$K_m = \oplus_{a_1+\cdots+a_n=m} S\Phi(a_1, \cdots, a_n).$$

Note: Throughout this chapter we will interpret $\Phi(a_1, \cdots, a_i - 1, \cdots, a_n) = 0$ if $a_i - 1$ is negative. Similarly, $\Phi(a_1, \cdots, a_i - 2, \cdots, a_n)$ and $\Phi(a_1, \cdots, a_i - 3, \cdots, a_n)$ will be zero if $a_i - 2$ and $a_i - 3$ are negative respectively.

For each $i,\ 1 \leq i \leq \theta$, let $\sigma_i, \tau_i : \mathbb{N} \to \mathbb{N}$ be the functions defined by

$$\sigma_i(a) = \begin{cases} 1, \text{ if } a \text{ is odd} \\ \\ N_i - 1, \text{ if } a \text{ is even,} \end{cases}$$

and $\tau_i(a) = \sum_{j=1}^{a} \sigma_i(j)$ for $a \geq 1, \tau_i(0) = 0$. For each $i,\ \theta + 1 \leq i \leq n$ we define $\sigma_i(a) = 1$ and $\tau_i(a) = a$.

We define the differential as follows:

$$d_i(\Phi(a_1, \cdots, a_n)) = \begin{cases} \prod_{i<l} (-1)^{a_l} q_{li}^{\sigma_i(a_i)\tau_l(a_l)} x_i^{\sigma_i(a_i)} \Phi(a_1, \cdots, a_i - 1, \cdots, a_n), & \text{if } a_i > 0 \\ \\ 0, & \text{if } a_i = 0 \end{cases}$$

Extend each d_i to an S-module homomorphism. We will now verify that $K_\bullet$ is a complex. Let $d = d_1 + \cdots + d_n$. Note that $x_i^{N_i} = 0$ when $i \leq \theta$ and $\sigma_i(a_i) + \sigma_i(a_i - 1) = N_i$. Consider,

$$\begin{aligned} & d_i d_i(\Phi(a_1, \cdots, a_n)) \\ & \qquad = \quad d_i \left(\left(\prod_{i<l} (-1)^{a_l} q_{li}^{\sigma_i(a_i)\tau_l(a_l)} \right) x_i^{\sigma_i(a_i)} \Phi(a_1, \cdots, a_i - 1, \cdots, a_n) \right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\prod_{i<l}(-1)^{a_l} q_{li}^{\sigma_i(a_i)\tau_l(a_l)}\right)\left(\prod_{i<m}(-1)^{a_m} q_{mi}^{\sigma_i(a_i-1)\tau_m(a_m)}\right) \\
&\qquad \cdot x_i^{\sigma_i(a_i)} x_i^{\sigma_i(a_i-1)} \Phi(a_1, \cdots, a_i - 2, \cdots, a_n) \\
&= 0
\end{aligned}
$$

Since for $i > \theta$, $a_i - 2$ is negative and in fact we get 0 by definition of d_i. If $i \leq \theta$, it is because $x_i^{N_i} = 0$.

If $i < j$, we have

$$
\begin{aligned}
&d_i d_j(\Phi(a_1, \cdots, a_n)) \\
&\quad = d_i\left(\left(\prod_{j<l}(-1)^{a_l} q_{lj}^{\sigma_j(a_j)\tau_l(a_l)}\right) x_j^{\sigma_j(a_j)} \Phi(a_1, \cdots, a_j - 1, \cdots, a_n)\right) \\
&\quad = \left(\prod_{j<l}(-1)^{a_l} q_{lj}^{\sigma_j(a_j)\tau_l(a_l)}\right)\left(\prod_{i<m}(-1)^{a_m} q_{mi}^{\sigma_i(a_i)\tau_m(a_m)}\right) \\
&\qquad \cdot x_j^{\sigma_j(a_j)} x_i^{\sigma_i(a_i)} \Phi(a_1, \cdots, a_i - 1, \cdots, a_j - 1, \cdots, a_n).
\end{aligned}
$$

$$
\begin{aligned}
&d_j d_i(\Phi(a_1, \cdots, a_n)) \\
&\quad = d_j\left(\left(\prod_{i<m}(-1)^{a_m} q_{mi}^{\sigma_i(a_i)\tau_m(a_m)}\right) x_i^{\sigma_i(a_i)} \Phi(a_1, \cdots, a_i - 1, \cdots, a_n)\right) \\
&\quad = \left(\prod_{i<m}(-1)^{a_m} q_{mi}^{\sigma_i(a_i)\tau_m(a_m)}\right)\left(\prod_{j<l}(-1)^{a_l} q_{lj}^{\sigma_j(a_j)\tau_l(a_l)}\right) \\
&\qquad \cdot x_i^{\sigma_i(a_i)} x_j^{\sigma_j(a_j)} \Phi(a_1, \cdots, a_i - 1, \cdots, a_j - 1, \cdots, a_n).
\end{aligned}
$$

Comparison shows that a scalar factor for the term in which $m = j$ changes from $(-1)^{a_j} q_{ji}^{\sigma_i(a_i)\tau_j(a_j)}$ to $(-1)^{a_j-1} q_{ji}^{\sigma_i(a_i)\tau_j(a_j-1)}$, and $x_j^{\sigma_j(a_j)} x_i^{\sigma_i(a_i)}$ is replaced by $x_j^{\sigma_j(a_j)} x_i^{\sigma_i(a_i)} = q_{ji}^{\sigma_i(a_i)\sigma_j(a_j)} x_i^{\sigma_i(a_i)} x_j^{\sigma_j(a_j)}$. Since $\tau_i(a_i) = \tau_i(a_i - 1) + \sigma_i(a_i)$, this illustrates that

$$d_i d_j + d_j d_i = 0.$$

Since $d^2 = 0$, we can say that $K_\bullet$ is indeed a complex.

Next we give a contracting homotopy to show that $K_\bullet$ is a resolution of k: Let $\eta \in S$, and fix $l, 1 \leq l \leq n$. Write

$$\eta = \begin{cases} \sum_{j=0}^{N_i-1} \eta_j x_l^j, \text{ if } 1 \leq l \leq \theta \\ \\ \sum_j \eta_j x_l^j, \text{ if } \theta+1 \leq l \leq n \end{cases}$$

where η_j is in the subalgebra of S generated by the x_i with $i \neq l$. Define

$s_l(\eta\Phi(a_1, \cdots, a_n))$

$$= \begin{cases} \displaystyle\sum_{j=0}^{N_i-1} s_l(\eta_j x_l^j \Phi(a_1, \cdots, a_n)), \text{ if } 1 \leq l \leq \theta \\ \\ \displaystyle\sum_j s_l(\eta_j x_l^j \Phi(a_1, \cdots, a_n)), \text{ if } \theta+1 \leq l \leq n \end{cases}$$

where

$s_l(\eta_j x_l^j \Phi(a_1, \cdots, a_n))$

$$= \begin{cases} \delta_{j>0}(\prod_{l<m}(-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)})\eta_j x_l^{j-1}\Phi(a_1, \cdots, a_l+1, \cdots, a_n), \\ \qquad\qquad\qquad\qquad \text{if } a_l \text{ is even with } 1 \leq l \leq \theta \\ \delta_{j,N_l-1}(\prod_{l<m}(-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)})\eta_j \Phi(a_1, \cdots, a_l+1, \cdots, a_n), \\ \qquad\qquad\qquad\qquad \text{if } a_l \text{ is odd with } 1 \leq l \leq \theta \\ \omega\eta_j x_l^{j-1}\Phi(a_1, \cdots, a_l+1, \cdots, a_n), \quad \text{if } \theta+1 \leq l \leq n \end{cases}$$

where $\delta_{j>0} = \begin{cases} 1, \text{ if } j>0 \\ 0, \text{ if } j=0 \end{cases}$ and $\omega = \dfrac{1}{\prod_{l<u}(-1)^{a_u} q_{ul}^{a_u}}$.

As explained below calculations show that for all $i,\ 1 \le i \le n$,

$$(s_i d_i + d_i s_i)(\eta_j x_i^j \Phi(a_1, \cdots, a_n)) = \begin{cases} \eta_j x_i^j \Phi(a_1, \cdots, a_n), & \text{if } j > 0 \text{ or } a_i > 0 \\ \\ 0, & \text{if } j = 0 \text{ and } a_i = 0 \end{cases}$$

The way we have defined our s_l and d_i, we get $s_l d_i + d_i s_l = 0$ for all i, l when $i \neq l$, as explained below. For each $x_1^{j_1} \cdots x_n^{j_n} \Phi(a_1, \cdots, a_n)$, let $C = c_{j_1, \cdots, j_n, a_1, \cdots, a_n}$ be the cardinality of the set of all $i(1 \le i \le n)$ such that both j_i and a_i are 0. Define

$$s(x_1^{j_1} \cdots x_n^{j_n} \Phi(a_1, \cdots, a_n)) = \begin{cases} \dfrac{1}{n - C}(s_1 + \cdots + s_n)(x_1^{j_1} \cdots x_n^{j_n} \Phi(a_1, \cdots, a_n)) \\ \\ 0, \quad \text{if } n = C \end{cases}$$

and since $d = d_1 + \cdots + d_n$, we have $sd + ds = id$ on each $K_m, m > 0$. That is, $K_\bullet$ is exact in positive degrees. For exactness at $K_0 = S$ we look at the kernel of the augmentation (counit) map $\varepsilon : S \to k$ and the image of $d(x_i^{j_i - 1} \cdots x_n^{j_n} \Phi(0, \cdots, 1, \cdots, 0))$. Observe that the kernel of ε is spanned over the field k by the elements $x_1^{j_1} \cdots x_\theta^{j_\theta} x_{\theta+1}^{j_{\theta+1}} \cdots x_n^{j_n} \cdot \Phi(0, \cdots, 0),\ 0 \le j_i \le N_i$, for $1 \le i \le \theta$ and $j_i \in \mathbb{N}$ for $\theta + 1 \le i \le n$, with at least one $j_i \neq 0$. Assume that $x_1^{j_1} \cdots x_\theta^{j_\theta} x_{\theta+1}^{j_{\theta+1}} \cdots x_n^{j_n} \Phi(0, \cdots, 0)$ is such an element, and assume that i is the smallest positive integer such that $j_i \neq 0$. Then

$$d(x_i^{j_i - 1} \cdots x_n^{j_n} \Phi(0, \cdots, 1, \cdots, 0)) = \vartheta x_i^{j_i} \cdots x_\theta^{j_\theta} x_{\theta+1}^{j_{\theta+1}} \cdots x_n^{j_n} \Phi(0, \cdots, 0)$$

where ϑ is a nonzero scalar. Thus $\ker(\varepsilon) = \operatorname{im}(d)$, and $K_\bullet$ is a free resolution of k as an S-module.

Now we provide below the explanation that $s_i d_i + d_i s_i = id$ and $s_l d_i + d_i s_l = 0$ for all i, l when $i \neq l$.

1) We first explain that $s_i d_i + d_i s_i = id$ for different cases.

a) For $\theta + 1 \leq i \leq n$, consider,

$$
\begin{aligned}
& (s_i d_i + d_i s_i)(\eta_j x_i^j \Phi(a_1, \cdots, a_n)) \\
= \;& s_i d_i(\eta_j x_i^j \Phi(a_1, \cdots, a_n)) + d_i s_i(\eta_j x_i^j \Phi(a_1, \cdots, a_n)) \\
= \;& s_i(\eta_j x_i^j \prod_{i<l} (-1)^{a_l} q_{li}^{a_l} \; x_i \Phi(a_1, \cdots, a_i - 1, \cdots, a_n)) \\
& \quad + d_i \left(\frac{1}{\prod_{i<u} (-1)^{a_u} q_{ui}^{a_u}} \eta_j x_i^{j-1} \Phi(a_1, \cdots, a_i + 1, \cdots, a_n) \right) \\
= \;& \prod_{i<l} (-1)^{a_l} q_{li}^{a_l} \frac{1}{\prod_{i<u} (-1)^{a_u} q_{ui}^{a_u}} \eta_j x_i^j \Phi(a_1, \cdots, a_i, \cdots, a_n) \\
& \quad + \frac{1}{\prod_{i<u} (-1)^{a_u} q_{ui}^{a_u}} \eta_j x_i^{j-1} \prod_{i<l} (-1)^{a_l} q_{li}^{a_l} x_i \Phi(a_1, \cdots, a_i, \cdots, a_n) \\
= \;& \eta_j x_i^j \Phi(a_1, \cdots, a_n)
\end{aligned}
$$

Since $a_i = 0$ or $a_i = 1$, we can see that only one term will survive in step 2. Therefore, at the end we are left with one term.

If $a_i = 0$ and $j = 0$ then $(s_i d_i + d_i s_i)(\eta_j x_i^j \Phi(a_1, \cdots, a_n)) = 0$ due to the expressions $a_i - 1$ and x_i^{j-1} in step 2.

b) For $1 \leq i \leq \theta$ with a_i even, consider,

$$
\begin{aligned}
& (s_i d_i + d_i s_i)(\eta_j x_i^j \Phi(a_1, \cdots, a_n)) \\
= \;& s_i d_i(\eta_j x_i^j \Phi(a_1, \cdots, a_n)) + d_i s_i(\eta_j x_i^j \Phi(a_1, \cdots, a_n)) \\
= \;& s_i(\eta_j x_i^j \prod_{i<l} (-1)^{a_l} q_{li}^{\sigma_i(a_i)\tau_l(a_l)} \; x_i^{\sigma_i(a_i)} \Phi(a_1, \cdots, a_i - 1, \cdots, a_n)) \\
& \quad + d_i(\delta_{j>0} \Big(\prod_{i<m} (-1)^{a_m} q_{mi}^{-\sigma_l(a_i+1)\tau_m(a_m)} \Big) \eta_j x_i^{j-1} \Phi(a_1, \cdots, a_i + 1, \cdots, a_n))
\end{aligned}
$$

$$
\begin{aligned}
= \;& s_i(\eta_j x_i^j \prod_{i<l}(-1)^{a_l} q_{li}^{\sigma_i(a_i)\tau_l(a_l)}\ x_i^{\sigma_i(a_i)} \Phi(a_1, \cdots, a_i - 1, \cdots, a_n)) \\
& +\delta_{j>0}\Big(\prod_{i<m}(-1)^{a_m} q_{mi}^{-\sigma_l(a_i+1)\tau_m(a_m)}\Big) \\
& \qquad\qquad \prod_{i<l}(-1)^{a_l} q_{li}^{\sigma_i(a_i+1)\tau_l(a_l)} \eta_j x_i^{(j-1)+\sigma_i(a_i+1)} \Phi(a_1, \cdots, a_i, \cdots, a_n) \\
= \;& \eta_j x_i^j \Phi(a_1, \cdots, a_n)
\end{aligned}
$$

Since a_i is even in the second term of step 3 we have $x_i^{(j-1)+\sigma_i(a_i+1)} = x_i^j$ and in the first term $x_i^{j+\sigma_i(a_i)} = x_i^{j+N_i-1} = 0$ because for $j > 0,\ j + N_i - 1 \geq N_i$.

If $a_i = 0$ and $j = 0$ then $(s_i d_i + d_i s_i)(\eta_j x_i^j \Phi(a_1, \cdots, a_n)) = 0$ due to the expressions $a_i - 1$ and x_i^{j-1} in step 2.

c) For $1 \leq i \leq \theta$ with a_i odd, consider,

$$
\begin{aligned}
& (s_i d_i + d_i s_i)(\eta_j x_i^j \Phi(a_1, \cdots, a_n)) \\
= \;& s_i d_i(\eta_j x_i^j \Phi(a_1, \cdots, a_n)) + d_i s_i(\eta_j x_i^j \Phi(a_1, \cdots, a_n)) \\
= \;& s_i(\eta_j x_i^j \prod_{i<l}(-1)^{a_l} q_{li}^{\sigma_i(a_i)\tau_l(a_l)}\ x_i^{\sigma_i(a_i)} \Phi(a_1, \cdots, a_i - 1, \cdots, a_n)) \\
& \qquad +d_i(\delta_{j,N_i-1}\Big(\prod_{i<m}(-1)^{a_m} q_{mi}^{-\sigma_l(a_i+1)\tau_m(a_m)}\Big)\eta_j \Phi(a_1, \cdots, a_i + 1, \cdots, a_n)) \\
= \;& s_i(\eta_j x_i^j \prod_{i<l}(-1)^{a_l} q_{li}^{\sigma_i(a_i)\tau_l(a_l)}\ x_i^{\sigma_i(a_i)} \Phi(a_1, \cdots, a_i - 1, \cdots, a_n)) \\
& \qquad +\delta_{j,N_i-1}\Big(\prod_{i<m}(-1)^{a_m} q_{mi}^{-\sigma_l(a_i+1)\tau_m(a_m)}\Big)\eta_j \\
& \qquad\qquad \prod_{i<l}(-1)^{a_l} q_{li}^{\sigma_i(a_i+1)\tau_l(a_l)} x_i^{\sigma_i(a_i+1)} \Phi(a_1, \cdots, a_i, \cdots, a_n) \\
= \;& \eta_j x_i^j \Phi(a_1, \cdots, a_n)
\end{aligned}
$$

Since a_i is odd in the second term of step 3 we have $x_i^{\sigma_i(a_i+1)} = x_i^{N_i-1}$ and for $j = N_i - 1,\ \delta_{j,N_i-1} = 1$ and in the first term $x_i^{j+\sigma_i(a_i)} = x_i^{N_i-1+1} = 0$.

If $a_i = 0$ and $j = 0$ then $(s_i d_i + d_i s_i)(\eta_j x_i^j \Phi(a_1, \cdots, a_n)) = 0$ due to the expressions $a_i - 1$ and x_i^{j-1} in step 2.

2) Now we explain that $s_l d_i + d_i s_l = 0$ for all i, l with $i \neq l$. For that we consider different cases.

Case $I : \theta + 1 \leq i, l \leq n$

a) If $l < i$, we have

$$\begin{aligned}
&(s_l d_i + d_i s_l)(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) \\
= \quad & s_l d_i(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) + d_i s_l(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) \\
= \quad & s_l(\eta_j x_l^j \prod_{i<m} (-1)^{a_m} q_{mi}^{a_m} \; x_i \Phi(a_1, \cdots, a_i - 1, \cdots, a_n)) \\
& \qquad + d_i \left(\frac{1}{\prod_{l<u} (-1)^{a_u} q_{ul}^{a_u}} \eta_j x_l^{j-1} \Phi(a_1, \cdots, a_l + 1, \cdots, a_n) \right) \\
= \quad & \prod_{i<m} (-1)^{a_m} q_{mi}^{a_m} \frac{1}{\prod_{l<u} (-1)^{a_u} q_{ul}^{a_u}} q_{li}^j \eta_j x_i x_l^{j-1} \\
& \qquad \qquad \qquad \cdot \Phi(a_1, \cdots, a_l + 1, \cdots, a_i - 1, \cdots, a_n) \\
& \qquad + \frac{1}{\prod_{l<u} (-1)^{a_u} q_{ul}^{a_u}} \eta_j x_l^{j-1} \prod_{i<m} (-1)^{a_m} q_{mi}^{a_m} x_i \\
& \qquad \qquad \qquad \cdot \Phi(a_1, \cdots, a_l + 1, \cdots, a_i - 1, \cdots, a_n)
\end{aligned}$$

Since $a_i, a_l \in \{0, 1\}$ the above sum will be zero if $a_l = 1$ or $a_i = 0$. So suppose not then when $u = i$ in the first term of step 3 we get $a_u = a_i - 1$ and in the second term we get $a_u = a_i$. Thus the terms will cancel each other due to the factor $(-1)^{a_u}$ in the second term.

b) If $i < l$, we have

$$\begin{aligned}
&(s_l d_i + d_i s_l)(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) \\
= \quad & s_l d_i(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) + d_i s_l(\eta_j x_l^j \Phi(a_1, \cdots, a_n))
\end{aligned}$$

$$\begin{aligned}
&= s_l(\eta_j x_l^j \prod_{i<m}(-1)^{a_m} q_{mi}^{a_m}\ x_i \Phi(a_1, \cdots, a_i - 1, \cdots, a_n)) \\
&\qquad +d_i\left(\frac{1}{\prod_{l<u}(-1)^{a_u} q_{ul}^{a_u}} \eta_j x_l^{j-1} \Phi(a_1, \cdots, a_l + 1, \cdots, a_n)\right) \\
&= \prod_{i<m}(-1)^{a_m} q_{mi}^{a_m} \frac{1}{\prod_{l<u}(-1)^{a_u} q_{ul}^{a_u}} q_{li}^j \eta_j x_i x_l^{j-1} \\
&\qquad\qquad \cdot\Phi(a_1 \cdots, a_i - 1, \cdots, a_l + 1, \cdots, a_n) \\
&\qquad +\frac{1}{\prod_{l<u}(-1)^{a_u} q_{ul}^{a_u}} \eta_j x_l^{j-1} \prod_{i<m}(-1)^{a_m} q_{mi}^{a_m} x_i \\
&\qquad\qquad \cdot\Phi(a_1, \cdots, a_i - 1, \cdots, a_l + 1, \cdots, a_n)
\end{aligned}$$

Since $a_i, a_l \in \{0, 1\}$ the above sum will be zero if $a_l = 1$ or $a_i = 0$. So suppose not then when $m = l$ in the first term of step 3 we get $a_m = a_l$ and in the second term we get $a_m = a_l + 1$. Thus the terms will cancel each other due to the factor $(-1)^{a_m}$ in the first term.

Case $II : 1 \le i, l \le \theta$

a) If $i < l$ with a_l even, we have

$$\begin{aligned}
&(s_l d_i + d_i s_l)(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) \\
&= s_l d_i(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) + d_i s_l(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) \\
&= s_l(\eta_j x_l^j \prod_{i<u}(-1)^{a_u} q_{ui}^{\sigma_i(a_i)\tau_u(a_u)} x_i^{\sigma_i(a_i)} \Phi(a_1, \cdots, a_i - 1, \cdots, a_n)) \\
&\qquad +d_i(\delta_{j>0}\Big(\prod_{l<m}(-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)}\Big) \eta_j x_l^{j-1} \Phi(a_1, \cdots, a_l + 1, \cdots, a_n)) \\
&= \prod_{i<u}(-1)^{a_u} q_{ui}^{\sigma_i(a_i)\tau_u(a_u)} \delta_{j>0} \prod_{l<m}(-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)} q_{li}^{j\sigma_i(a_i)} \eta_j x_i^{\sigma_i(a_i)} x_l^{j-1} \\
&\qquad\qquad \cdot\Phi(a_1, \cdots, a_i - 1, \cdots, a_l + 1, \cdots, a_n) \\
&\qquad +\delta_{j>0} \prod_{l<m}(-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)} \eta_j x_l^{j-1} \prod_{i<u}(-1)^{a_u} q_{ui}^{\sigma_i(a_i)\tau_u(a_u)} x_i^{\sigma_i(a_i)} \\
&\qquad\qquad \cdot\Phi(a_1, \cdots, a_i - 1, \cdots, a_l + 1, \cdots, a_n)
\end{aligned}$$

If $a_l = 1$ or $a_i = 0$ then we get zero. So suppose not then when $u = l$ in the first term of step 3 we get $a_u = a_l$ and in the second term we get $a_u = a_l + 1$. Also a_l is even and in the second term $\tau_u(a_u) = \tau_l(a_l + 1) = \tau_l(a_l) + 1$. Thus the terms will cancel each other.

b) If $i < l$ with a_l odd, we have

$$\begin{aligned}
&(s_l d_i + d_i s_l)(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) \\
&= s_l d_i(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) + d_i s_l(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) \\
&= s_l(\eta_j x_l^j \prod_{i<u} (-1)^{a_u} q_{ui}^{\sigma_i(a_i)\tau_u(a_u)} x_i^{\sigma_i(a_i)} \Phi(a_1, \cdots, a_i - 1, \cdots, a_n)) \\
&\qquad + d_i(\delta_{j,N_l-1} \Big(\prod_{l<m} (-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)} \Big) \eta_j \Phi(a_1, \cdots, a_l + 1, \cdots, a_n)) \\
&= \prod_{i<u} (-1)^{a_u} q_{ui}^{\sigma_i(a_i)\tau_u(a_u)} \delta_{j,N_l-1} \prod_{l<m} (-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)} q_{li}^{j\sigma_i(a_i)} \eta_j x_i^{\sigma_i(a_i)} \\
&\qquad\qquad \cdot \Phi(a_1, \cdots, a_i - 1, \cdots, a_l + 1, \cdots, a_n) \\
&\qquad + \delta_{j,N_l-1} \prod_{l<m} (-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)} \prod_{i<u} (-1)^{a_u} q_{ui}^{\sigma_i(a_i)\tau_u(a_u)} \eta_j x_i^{\sigma_i(a_i)} \\
&\qquad\qquad \cdot \Phi(a_1, \cdots, a_i - 1, \cdots, a_l + 1, \cdots, a_n)
\end{aligned}$$

If $a_l = 1$ or $a_i = 0$ then we get zero. So suppose not then when $u = l$ in the first term of step 3 we get $a_u = a_l$ and in the second term we get $a_u = a_l + 1$. Also a_l is odd and in the second term $\tau_u(a_u) = \tau_l(a_l + 1) = \tau_l(a_l) + \sigma_l(a_l + 1) = \tau_l(a_l) + N_l - 1$ and for $j = N_l - 1$, $\delta_{j,N_l-1} = 1$. Thus the terms will cancel each other.

c) If $l < i$ with a_l even, we have

$$\begin{aligned}
&(s_l d_i + d_i s_l)(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) \\
&= s_l d_i(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) + d_i s_l(\eta_j x_l^j \Phi(a_1, \cdots, a_n))
\end{aligned}$$

$$
\begin{aligned}
&= s_l(\eta_j x_l^j \prod_{i<u}(-1)^{a_u} q_{ui}^{\sigma_i(a_i)\tau_u(a_u)} x_i^{\sigma_i(a_i)} \Phi(a_1,\cdots,a_i-1,\cdots,a_n)) \\
&\qquad + d_i(\delta_{j>0}\Big(\prod_{l<m}(-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)}\Big)\eta_j x_l^{j-1}\Phi(a_1,\cdots,a_l+1,\cdots,a_n)) \\
&= \prod_{i<u}(-1)^{a_u} q_{ui}^{\sigma_i(a_i)\tau_u(a_u)} \delta_{j>0} \prod_{l<m}(-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)} q_{li}^{j\sigma_i(a_i)} \eta_j x_i^{\sigma_i(a_i)} x_l^{j-1} \\
&\qquad\qquad \cdot\Phi(a_1,\cdots,a_l+1,\cdots,a_i-1,\cdots,a_n) \\
&\qquad + \delta_{j>0}\prod_{l<m}(-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)} \eta_j x_l^{j-1} \prod_{i<u}(-1)^{a_u} q_{ui}^{\sigma_i(a_i)\tau_u(a_u)} x_i^{\sigma_i(a_i)} \\
&\qquad\qquad \cdot\Phi(a_1,\cdots,a_l+1,\cdots,a_i-1,\cdots,a_n)
\end{aligned}
$$

If $a_l = 1$ or $a_i = 0$ then we get zero. So suppose not then when $m = i$ in the first term of step 3 we get $a_m = a_i - 1$ and in the second term we get $a_m = a_i$. Also a_l is even and in the first term $\tau_m(a_m) = \tau_i(a_i - 1) = \tau_i(a_i) - \sigma_i(a_i)$. Thus the terms will cancel each other.

d) If $l < i$ with a_l odd, we have

$$
\begin{aligned}
&(s_l d_i + d_i s_l)(\eta_j x_l^j \Phi(a_1,\cdots,a_n)) \\
&= s_l d_i(\eta_j x_l^j \Phi(a_1,\cdots,a_n)) + d_i s_l(\eta_j x_l^j \Phi(a_1,\cdots,a_n)) \\
&= s_l(\eta_j x_l^j \prod_{i<u}(-1)^{a_u} q_{ui}^{\sigma_i(a_i)\tau_u(a_u)} x_i^{\sigma_i(a_i)} \Phi(a_1,\cdots,a_i-1,\cdots,a_n)) \\
&\qquad + d_i(\delta_{j,N_l-1}\Big(\prod_{l<m}(-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)}\Big)\eta_j \Phi(a_1,\cdots,a_l+1,\cdots,a_n)) \\
&= \prod_{i<u}(-1)^{a_u} q_{ui}^{\sigma_i(a_i)\tau_u(a_u)} \delta_{j,N_l-1} \prod_{l<m}(-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)} q_{li}^{j\sigma_i(a_i)} \eta_j x_i^{\sigma_i(a_i)} \\
&\qquad\qquad \cdot\Phi(a_1,\cdots,a_l+1,\cdots,a_i-1,\cdots,a_n) \\
&\qquad + \delta_{j,N_l-1}\prod_{l<m}(-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)} \prod_{i<u}(-1)^{a_u} q_{ui}^{\sigma_i(a_i)\tau_u(a_u)} \eta_j x_i^{\sigma_i(a_i)} \\
&\qquad\qquad \cdot\Phi(a_1,\cdots,a_l+1,\cdots,a_i-1,\cdots,a_n)
\end{aligned}
$$

If $a_l = 1$ or $a_i = 0$ then we get zero. So suppose not then when $m = i$ in the first term of step 3 we get $a_m = a_i - 1$ and in the second term we get $a_m = a_i$. Also a_l is odd and in the first term $\tau_m(a_m) = \tau_i(a_i - 1) = \tau_i(a_i) - \sigma_i(a_i)$ and for $j = N_l - 1,\ \delta_{j,N_l-1} = 1$. Thus the terms will cancel each other.

3) Finally we explain that $s_l d_i + d_i s_l = 0$ for the mixed case with $i \neq l$.

a) Consider the case when $l < i$ with $1 \leq l \leq \theta$ and $\theta + 1 \leq i \leq n$ and a_l is even. Then

$$
\begin{aligned}
&(s_l d_i + d_i s_l)(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) \\
= \;& s_l d_i(\eta_j x_l^j \Phi(a_1, \cdots, a_\theta, a_{\theta+1}, \cdots, a_n)) + d_i s_l(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) \\
= \;& s_l(\eta_j x_l^j \prod_{i<u} (-1)^{a_u} q_{ui}^{a_u} x_i \Phi(a_1, \cdots, a_i - 1, \cdots, a_n)) \\
& \quad + d_i(\delta_{j>0}(\prod_{l<m} (-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)}) \eta_j x_l^{j-1} \Phi(a_1, \cdots, a_l + 1, \cdots, a_n)) \\
= \;& \delta_{j>0}(\prod_{l<m} (-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)}) \\
& \qquad \prod_{i<u} (-1)^{a_u} q_{ui}^{a_u} q_{li}^j x_i \eta_j x_l^{j-1} \Phi(a_1, \cdots, a_l + 1, \cdots, a_i - 1, \cdots, a_n) \\
& + \delta_{j>0}(\prod_{l<m} (-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)}) \eta_j x_l^{j-1} \\
& \qquad \prod_{i<u} (-1)^{a_u} q_{ui}^{a_u} x_i \Phi(a_1, \cdots, a_l + 1, \cdots, a_i - 1, \cdots, a_n)
\end{aligned}
$$

If $a_i = 0$, we get zero. So suppose $a_i \neq 0$ which implies $a_i = 1$. When $m = i$ in the first term of step 3 we get $a_m = a_i - 1$ and in the second term we get $a_m = a_i$ and also a_l is even. Thus the terms will cancel each other.

b) Suppose a_l is odd. Then

$$
\begin{aligned}
&(s_l d_i + d_i s_l)(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) \\
= \;& s_l d_i(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) + d_i s_l(\eta_j x_l^j \Phi(a_1, \cdots, a_n))
\end{aligned}
$$

$$
\begin{aligned}
&= s_l(\eta_j x_l^j \prod_{i<u} (-1)^{a_u} q_{ui}^{a_u} x_i \Phi(a_1, \cdots, a_i - 1, \cdots, a_n)) \\
&\qquad + d_i(\delta_{j,N_l-1}(\prod_{l<m} (-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)}) \eta_j \Phi(a_1, \cdots, a_l + 1, \cdots, a_n)) \\
&= \delta_{j,N_l-1}(\prod_{l<m} (-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)}) q_{li}^j \eta_j \\
&\qquad\qquad \prod_{i<u} (-1)^{a_u} q_{ui}^{a_u} x_i \Phi(a_1, \cdots, a_l + 1, \cdots, a_i - 1, \cdots, a_n) \\
&\quad + \delta_{j,N_l-1}(\prod_{l<m} (-1)^{a_m} q_{ml}^{-\sigma_l(a_l+1)\tau_m(a_m)}) \eta_j \\
&\qquad\qquad \prod_{i<u} (-1)^{a_u} q_{ui}^{a_u} x_i \Phi(a_1, \cdots, a_l + 1, \cdots, a_i - 1, \cdots, a_n)
\end{aligned}
$$

If $a_i = 0$ or $j \neq N_l - 1$, we get zero. So suppose $j = N_l - 1$ and $a_i \neq 0$ which implies $a_i = 1$. When $m = i$ in the first term of step 3 we get $a_m = a_i - 1$ and in the second term we get $a_m = a_i$ and also a_l is odd. Thus the terms will cancel each other.

c) Consider the case when $1 \leq i \leq \theta$ and $\theta + 1 \leq l \leq n$. Then

$$
\begin{aligned}
&(s_l d_i + d_i s_l)(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) \\
&= s_l d_i(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) + d_i s_l(\eta_j x_l^j \Phi(a_1, \cdots, a_n)) \\
&= s_l(\eta_j x_l^j \prod_{i<m} (-1)^{a_m} q_{mi}^{\sigma_i(a_i)\tau_m(a_m)} x_i^{\sigma_i(a_i)} \Phi(a_1, \cdots, a_i - 1, \cdots, a_n)) \\
&\qquad + d_i \left(\frac{1}{\prod_{l<u} (-1)^{a_u} q_{ul}^{a_u}} \eta_j x_l^{j-1} \Phi(a_1, \cdots, a_l + 1, \cdots, a_n) \right) \\
&= \prod_{i<m} (-1)^{a_m} q_{mi}^{\sigma_i(a_i)\tau_m(a_m)} q_{li}^{j\sigma_i(a_i)} \\
&\qquad\qquad \frac{1}{\prod_{l<u} (-1)^{a_u} q_{ul}^{a_u}} \eta_j x_i^{\sigma_i(a_i)} x_l^{j-1} \Phi(a_1, \cdots, a_i - 1, \cdots, a_l + 1, \cdots, a_n) \\
&+ \frac{1}{\prod_{l<u} (-1)^{a_u} q_{ul}^{a_u}} \eta_j x_l^{j-1} \prod_{i<m} (-1)^{a_m} q_{mi}^{\sigma_i(a_i)\tau_m(a_m)} x_i^{\sigma_i(a_i)} \\
&\qquad\qquad \cdot \Phi(a_1, \cdots, a_i - 1, \cdots, a_l + 1, \cdots, a_n)
\end{aligned}
$$

If $a_i = 0$ or $a_l = 1$ we get zero. So suppose $a_i \neq 0$ and $a_l \neq 1$ which implies $a_l = 0$

then when $m = l$ in the first term of step 3 we get $a_m = a_l$ and in the second term we get $a_m = a_l + 1$. Thus the terms will cancel each other due to the factor $(-1)^{a_m}$ in the second term.

Next to compute $Ext_S^*(k,k)$ we apply $Hom_S(-,k)$ to $K_\bullet$. The $Hom_S(-,k)$ functor induces the differential d^*. Moreover it is the zero map since $x_i^{\sigma_i(a_i)}$ is always in the augmentation ideal. Thus the cohomology is the complex $Hom_S(K_\bullet, k)$. Now let $\eta_i \in Hom_S(K_1, k)$ be the function dual to $\Phi(0, \cdots, 0, 1, 0 \cdots, 0)$ (the 1 in the ith position) and $\xi_i \in Hom_S(K_2, k)$ for $i \leq \theta$ be the function dual to $\Phi(0, \cdots, 0, 2, 0 \cdots, 0)$ (the 2 in the ith position). By abusing the notation we will identify the functions ξ_i, η_i with the corresponding elements in $H^2(S,k)$ and $H^1(S,k)$, respectively. Further, we will show that they generate $H^*(S,k)$ and determine the relations among them. To do this we denote by ξ_i and η_i the corresponding chain maps $\xi_i : K_\bullet \to K_{\bullet-2}$ and $\eta_i : K_\bullet \to K_{\bullet-1}$ defined by

$$\xi_i(\Phi(a_1, \cdots, a_n)) = \prod_{l<i} q_{il}^{N_i \tau_l(a_l)} \Phi(a_1, \cdots, a_i - 2, \cdots, a_n), \quad \text{if } 1 \leq i \leq \theta$$

$$\eta_i(\Phi(a_1, \cdots, a_n)) = \prod_{i<l} q_{li}^{(\sigma_i(a_i)-1)\tau_l(a_l)} \prod_{l<i} (-1)^{a_l} q_{il}^{\tau_l(a_l)} x_i^{\sigma_i(a_i)-1} \Phi(a_1, \cdots, a_i - 1, \cdots, a_n)$$

To show that ξ_i and η_i are indeed chain maps, consider,

$$
\begin{aligned}
&\xi_i d(\Phi(a_1, \cdots, a_n)) \\
&\quad = \quad \xi_i(d_1 + d_2 \cdots + d_i + \cdots + d_{n-1} + d_n)(\Phi(a_1, \cdots, a_i, \cdots, a_n)) \\
&\quad = \quad \xi_i d_1(\Phi(a_1, \cdots, a_i, \cdots, a_n)) + \xi_i d_2(\Phi(a_1, \cdots, a_i, \cdots, a_n)) + \cdots \\
&\qquad\qquad + \xi_i d_i(\Phi(a_1, \cdots, a_i, \cdots, a_n)) + \cdots + \xi_i d_{n-1}(\Phi(a_1, \cdots, a_i, \cdots, a_n)) \\
&\qquad\qquad\qquad + \xi_i d_n(\Phi(a_1, \cdots, a_i, \cdots, a_n)) \\
&\quad = \quad \xi_i \left(\prod_{l=2}^{n} (-1)^{a_l} q_{l1}^{\sigma_1(a_1)\tau_l(a_l)} x_1^{\sigma_1(a_1)} (\Phi(a_1 - 1, \cdots, a_n)) \right) \\
&\qquad\qquad + \xi_i \left(\prod_{l=3}^{n} (-1)^{a_l} q_{l2}^{\sigma_2(a_2)\tau_l(a_l)} x_2^{\sigma_2(a_2)} (\Phi(a_1, a_2 - 1 \cdots, a_n)) \right) + \cdots \\
&\qquad\qquad\qquad + \xi_i \left(\prod_{l=i+1}^{n} (-1)^{a_l} q_{li}^{\sigma_i(a_i)\tau_l(a_l)} x_i^{\sigma_i(a_i)} (\Phi(a_1, \cdots, a_i - 1, \cdots, a_n)) \right) + \cdots \\
&\qquad\qquad\qquad\quad + \xi_i \left((-1)^{a_n} q_{ni}^{\sigma_{n-1}(a_{n-1})\tau_n(a_n)} x_{n-1}^{\sigma_{n-1}(a_{n-1})} (\Phi(a_1, \cdots, a_{n-1} - 1, a_n)) \right) \\
&\qquad\qquad\qquad\qquad + \xi_i \left(x_n^{\sigma_n(a_n)} (\Phi(a_1, \cdots, a_n - 1)) \right) \\
&\quad = \quad \prod_{l=2}^{n} (-1)^{a_l} q_{l1}^{\sigma_1(a_1)\tau_l(a_l)} x_1^{\sigma_1(a_1)} \prod_{m=1}^{i-1} q_{im}^{N_i \tau_m(a_m)} \Phi(a_1 - 1, \cdots, a_i - 2, \cdots, a_n) \\
&\quad + \quad \prod_{l=3}^{n} (-1)^{a_l} q_{l2}^{\sigma_2(a_2)\tau_l(a_l)} x_2^{\sigma_2(a_2)} \prod_{m=1}^{i-1} q_{im}^{N_i \tau_m(a_m)} \Phi(a_1, a_2 - 1 \cdots, a_i - 2, \cdots, a_n) + \cdots \\
&\quad + \quad \prod_{l=i+1}^{n} (-1)^{a_l} q_{li}^{\sigma_i(a_i)\tau_l(a_l)} x_i^{\sigma_i(a_i)} \prod_{m=1}^{i-1} q_{im}^{N_i \tau_m(a_m)} \Phi(a_1, \cdots, a_i - 3, \cdots, a_n) + \cdots \\
&\quad + \quad (-1)^{a_n} q_{ni}^{\sigma_{n-1}(a_{n-1})\tau_n(a_n)} x_{n-1}^{\sigma_{n-1}(a_{n-1})} \prod_{m=1}^{i-1} q_{im}^{N_i \tau_m(a_m)} \Phi(a_1, \cdots, a_i - 2, \cdots, a_{n-1} - 1, a_n) \\
&\quad + \quad x_n^{\sigma_n(a_n)} \prod_{m=1}^{i-1} q_{im}^{N_i \tau_m(a_m)} \Phi(a_1, \cdots, a_i - 2, \cdots, a_n - 1)
\end{aligned}
$$

In the first term of last the expression when $l = i$ we have $\tau_l(a_l) = \tau_i(a_i) = \tau_i(a_i - 2) + \sigma_i(a_i - 1) + \sigma_i(a_i) = \tau_i(a_i - 2) + N_i$ and for $m = 1$, $\tau_m(a_m) = \tau_1(a_1 - 1)$.

Next consider,

$$
\begin{aligned}
& d\xi_i(\Phi(a_1, \cdots, a_n)) \\
&= (d_1 + \cdots + d_i + \cdots + d_n)\xi_i(\Phi(a_1, \cdots, a_i, \cdots, a_n)) \\
&= d_1\xi_i(\Phi(a_1, \cdots, a_i, \cdots, a_n)) + \cdots + d_i\xi_i(\Phi(a_1, \cdots, a_i, \cdots, a_n)) + \cdots \\
&\qquad + d_n\xi_i(\Phi(a_1, \cdots, a_i, \cdots, a_n)) \\
&= d_1\left(\prod_{m=1}^{i-1} q_{im}^{N_i\tau_m(a_m)}\Phi(a_1, \cdots, a_i - 2, \cdots, a_n)\right) + \cdots \\
&\qquad + d_i\left(\prod_{m=1}^{i-1} q_{im}^{N_i\tau_m(a_m)}\Phi(a_1, \cdots, a_i - 2, \cdots, a_n)\right) + \cdots \\
&\qquad\qquad + d_n\left(\prod_{m=1}^{i-1} q_{im}^{N_i\tau_m(a_m)}\Phi(a_1, \cdots, a_i - 2, \cdots, a_n)\right) \\
&= \prod_{m=1}^{i-1} q_{im}^{N_i\tau_m(a_m)} \prod_{l=2}^{n}(-1)^{a_l} q_{l1}^{\sigma_1(a_1)\tau_l(a_l)} x_1^{\sigma_1(a_1)}\Phi(a_1 - 1, \cdots, a_i - 2, \cdots, a_n) + \cdots \\
&+ \prod_{m=1}^{i-1} q_{im}^{N_i\tau_m(a_m)} \prod_{l=i+1}^{n}(-1)^{a_l} q_{li}^{\sigma_i(a_i-2)\tau_l(a_l)} x_i^{\sigma_i(a_i-2)}\Phi(a_1, \cdots, a_i - 3, \cdots, a_n) + \cdots \\
&+ \prod_{m=1}^{i-1} q_{im}^{N_i\tau_m(a_m)} x_n^{\sigma_n(a_n)}\Phi(a_1, \cdots, a_i - 2, \cdots, a_n - 1)
\end{aligned}
$$

In the first term of the last expression when $l = i$ we have $\tau_l(a_l) = \tau_i(a_i - 2)$ and for $m = 1$, $\tau_m(a_m) = \tau_1(a_1) = \tau_1(a_1 - 1) + \sigma_1(a_1)$. Also note that $\sigma_i(a_i) = \sigma_i(a_i - 2)$.

Hence, by comparing both the expressions, we see that $\xi_i d = d\xi_i$. This proves that ξ_i is a chain map.

Similarly, we will prove that η_i is a chain map. For that first consider,

$$
\begin{aligned}
&\eta_i d(\Phi(a_1, \cdots, a_n)) \\
&= \eta_i(d_1 + \cdots + d_i + \cdots + d_n)(\Phi(a_1, \cdots, a_i, \cdots, a_n)) \\
&= \eta_i d_1(\Phi(a_1, \cdots, a_i, \cdots, a_n)) + \cdots + \eta_i d_i(\Phi(a_1, \cdots, a_i, \cdots, a_n)) + \cdots \\
&\qquad + \eta_i d_n(\Phi(a_1, \cdots, a_i, \cdots, a_n)) \\
&= \eta_i \left(\prod_{l=2}^{n} (-1)^{a_l} q_{l1}^{\sigma_1(a_1)\tau_l(a_l)} x_1^{\sigma_1(a_1)} (\Phi(a_1 - 1, \cdots, a_n)) \right) + \cdots \\
&\qquad + \eta_i \left(\prod_{l=i+1}^{n} (-1)^{a_l} q_{li}^{\sigma_i(a_i)\tau_l(a_l)} x_i^{\sigma_i(a_i)} (\Phi(a_1, \cdots, a_i - 1, \cdots, a_n)) \right) + \cdots \\
&\qquad + \eta_i \left(x_n^{\sigma_n(a_n)} (\Phi(a_1, \cdots, a_n - 1)) \right) \\
&= \prod_{l=2}^{n} (-1)^{a_l} q_{l1}^{\sigma_1(a_1)\tau_l(a_l)} x_1^{\sigma_1(a_1)} \prod_{m=i+1}^{n} q_{mi}^{(\sigma_i(a_i)-1)\tau_m(a_m)} \\
&\qquad \prod_{m=1}^{i-1} (-1)^{a_m} q_{im}^{\tau_m(a_m)} x_i^{\sigma_i(a_i)-1} \Phi(a_1 - 1, \cdots, a_i - 1, \cdots, a_n) + \cdots \\
&+ \prod_{l=i+1}^{n} (-1)^{a_l} q_{li}^{\sigma_i(a_i)\tau_l(a_l)} x_i^{\sigma_i(a_i)} \prod_{m=i+1}^{n} q_{mi}^{(\sigma_i(a_i-1)-1)\tau_m(a_m)} \\
&\qquad \prod_{m=1}^{i-1} (-1)^{a_m} q_{im}^{\tau_m(a_m)} x_i^{\sigma_i(a_i-1)-1} \Phi(a_1, \cdots, a_i - 2, \cdots, a_n) + \cdots \\
&+ x_n^{\sigma_n(a_n)} \prod_{m=i+1}^{n} q_{mi}^{(\sigma_i(a_i)-1)\tau_m(a_m)} \prod_{m=1}^{i-1} (-1)^{a_m} q_{im}^{\tau_m(a_m)} x_i^{\sigma_i(a_i)-1} \\
&\qquad \cdot \Phi(a_1, \cdots, a_i - 1, \cdots, a_n - 1)
\end{aligned}
$$

In the first term of the last expression when $m = 1$ we have $\tau_m(a_m) = \tau_1(a_1 - 1)$ and for $l = i$, $\tau_l(a_l) = \tau_i(a_i) = \tau_i(a_i - 1) + \sigma_i(a_i)$. In the third term when $m = n$, $\tau_m(a_m) = \tau_n(a_n - 1)$.

Now consider,

$$d\eta_i(\Phi(a_1, \cdots, a_n))$$

$$
\begin{aligned}
&= d_1\eta_i(\Phi(a_1,\cdots,a_i,\cdots,a_n)) + \cdots + d_i\eta_i(\Phi(a_1,\cdots,a_i,\cdots,a_n)) + \cdots \\
&\qquad + d_n\eta_i(\Phi(a_1,\cdots,a_i,\cdots,a_n)) \\
&= d_1\left(\prod_{m=i+1}^{n} q_{mi}^{(\sigma_i(a_i)-1)\tau_m(a_m)} \prod_{m=1}^{i-1}(-1)^{a_m} q_{im}^{\tau_m(a_m)} x_i^{\sigma_i(a_i)-1}\Phi(a_1,\cdots,a_i-1,\cdots,a_n)\right) + \cdots \\
&+ d_i\left(\prod_{m=i+1}^{n} q_{mi}^{(\sigma_i(a_i)-1)\tau_m(a_m)} \prod_{m=1}^{i-1}(-1)^{a_m} q_{im}^{\tau_m(a_m)} x_i^{\sigma_i(a_i)-1}\Phi(a_1,\cdots,a_i-1,\cdots,a_n)\right) + \cdots \\
&+ d_n\left(\prod_{m=i+1}^{n} q_{mi}^{(\sigma_i(a_i)-1)\tau_m(a_m)} \prod_{m=1}^{i-1}(-1)^{a_m} q_{im}^{\tau_m(a_m)} x_i^{\sigma_i(a_i)-1}\Phi(a_1,\cdots,a_i-1,\cdots,a_n)\right) \\
&= \prod_{m=i+1}^{n} q_{mi}^{(\sigma_i(a_i)-1)\tau_m(a_m)} \prod_{m=1}^{i-1}(-1)^{a_m} q_{im}^{\tau_m(a_m)} x_i^{\sigma_i(a_i)-1} \\
&\qquad\qquad \prod_{l=2}^{n}(-1)^{a_l} q_{l1}^{\sigma_1(a_1)\tau_l(a_l)} x_1^{\sigma_1(a_1)}\Phi(a_1-1,\cdots,a_i-1,\cdots,a_n) + \cdots \\
&+ \prod_{m=i+1}^{n} q_{mi}^{(\sigma_i(a_i)-1)\tau_m(a_m)} \prod_{m=1}^{i-1}(-1)^{a_m} q_{im}^{\tau_m(a_m)} x_i^{\sigma_i(a_i)-1} \\
&\qquad\qquad \prod_{l=i+1}^{n}(-1)^{a_l} q_{li}^{\sigma_i(a_i-1)\tau_l(a_l)} x_i^{\sigma_i(a_i-1)}\Phi(a_1,\cdots,a_i-2,\cdots,a_n) + \cdots \\
&+ \prod_{m=i+1}^{n} q_{mi}^{(\sigma_i(a_i)-1)\tau_m(a_m)} \prod_{m=1}^{i-1}(-1)^{a_m} q_{im}^{\tau_m(a_m)} x_i^{\sigma_i(a_i)-1} x_n^{\sigma_n(a_n)}\Phi(a_1,\cdots,a_i-1,\cdots,a_n-1)
\end{aligned}
$$

In the first term of the last expression when $m = 1$ we have $\tau_m(a_m) = \tau_1(a_1) = \tau_1(a_1 - 1) + \sigma_1(a_1)$ and for $l = i$, $\tau_l(a_l)$ is replaced by $\tau_i(a_i - 1)$ since here a_i term is replaced by $a_i - 1$ term and $x_i^{\sigma_i(a_i)-1}x_1^{\sigma_1(a_1)} = q_{i1}^{\sigma_1(a_1)(\sigma_i(a_i)-1)} \cdot x_1^{\sigma_1(a_1)}x_i^{\sigma_i(a_i)-1}$. In the third term when $m = n$, $\tau_m(a_m) = \tau_n(a_n) = \tau_n(a_n - 1) + \sigma_n(a_n)$.

Therefore, by comparing both the expressions, we see that $\eta_i d = d\eta_i$ proving that η_i is a chain map.

Theorem III.1. *Let S be the k-algebra generated by $x_1, \cdots, x_n$, subject to relations $x_ix_j = q_{ij}x_jx_i$ for all $i < j$, $x_i^{N_i} = 0$ for $1 \le i \le \theta$. Then $H^*(S, k)$ is generated by ξ_i $(i = 1, \cdots, \theta)$ and η_i $(i = 1, \cdots, n)$ where deg $\xi_i = 2$ and deg $\eta_i = 1$, subject to the* *relations*

$$\xi_i\xi_j = q_{ji}^{N_iN_j}\xi_j\xi_i,\ \ \eta_i\xi_j = q_{ji}^{N_j}\xi_j\eta_i, \quad and \ \ \eta_i\eta_j = -q_{ji}\eta_j\eta_i.$$

Proof. The ring structure of the subalgebra of $H^*(S,k)$ generated by ξ_i, η_i is given by composition of these chain maps.

We will first show that the relations hold. If $i < j$ we have,

$$\begin{aligned}
\xi_i\xi_j(\Phi(a_1,\cdots,a_n)) &= \xi_i\left(\prod_{l<j} q_{jl}^{N_j\tau_l(a_l)}\Phi(a_1,\cdots,a_j-2,\cdots,a_n)\right)\\
&= \prod_{l<j} q_{jl}^{N_j\tau_l(a_l)}\prod_{m<i} q_{im}^{N_i\tau_m(a_m)}\Phi(a_1,\cdots,a_i-2,\cdots,a_j-2,\cdots,a_n)
\end{aligned}$$

$$\begin{aligned}
\xi_j\xi_i(\Phi(a_1,\cdots,a_n)) &= \xi_j\left(\prod_{m<i} q_{im}^{N_i\tau_m(a_m)}\Phi(a_1,\cdots,a_i-2,\cdots,a_n)\right)\\
&= \prod_{m<i} q_{im}^{N_i\tau_m(a_m)}\prod_{l<j} q_{jl}^{N_j\tau_l(a_l)}\Phi(a_1,\cdots,a_i-2,\cdots,a_j-2,\cdots,a_n)
\end{aligned}$$

Comparison shows that a scalar factor for the term in which $l = i$ changes from $q_{ji}^{N_j\tau_i(a_i)}$ to $q_{ji}^{N_j\tau_i(a_i-2)}$. Since $\tau_i(a_i) = \tau_i(a_i-2)+\sigma_i(a_i-1)+\sigma_i(a_i) = \tau_i(a_i-2)+N_i$, this shows that

$$\xi_i\xi_j = q_{ji}^{N_iN_j}\xi_j\xi_i.$$

Next, consider,

$$\begin{aligned}
&\eta_i\xi_j(\Phi(a_1,\cdots,a_n))\\
&\quad = \ \eta_i\left(\prod_{l<j} q_{jl}^{N_j\tau_l(a_l)}\Phi(a_1,\cdots,a_j-2,\cdots,a_n)\right)\\
&\quad = \ \prod_{l<j} q_{jl}^{N_j\tau_l(a_l)}\prod_{i<m} q_{mi}^{(\sigma_i(a_i)-1)\tau_m(a_m)}\prod_{m<i}(-1)^{a_m}q_{im}^{\tau_m(a_m)}x_i^{\sigma_i(a_i)-1}\\
&\qquad\qquad \cdot\Phi(a_1,\cdots,a_i-1,\cdots,a_j-2,\cdots,a_n)
\end{aligned}$$

$$
\begin{aligned}
&\xi_j\eta_i(\Phi(a_1,\cdots,a_n)) \\
&\quad = \quad \xi_j\left(\prod_{i<m} q_{mi}^{(\sigma_i(a_i)-1)\tau_m(a_m)}\prod_{m<i}(-1)^{a_m}q_{im}^{\tau_m(a_m)}x_i^{\sigma_i(a_i)-1}\Phi(a_1,\cdots,a_i-1,\cdots,a_n)\right) \\
&\quad = \quad \prod_{i<m} q_{mi}^{(\sigma_i(a_i)-1)\tau_m(a_m)}\prod_{m<i}(-1)^{a_m}q_{im}^{\tau_m(a_m)}x_i^{\sigma_i(a_i)-1}\prod_{l<j} q_{jl}^{N_j\tau_l(a_l)} \\
&\qquad\qquad \cdot\Phi(a_1,\cdots,a_i-1,\cdots,a_j-2,\cdots,a_n)
\end{aligned}
$$

Comparison shows that a scalar factor for the term in which $l=i$ changes from $q_{ji}^{N_j\tau_i(a_i)}$ to $q_{ji}^{N_j\tau_i(a_i-1)}$, and a scalar factor for the term in which $m=j$ changes from $q_{ji}^{(\sigma_i(a_i)-1)\tau_j(a_j)}$ to $q_{ji}^{(\sigma_i(a_i)-1)\tau_j(a_j-2)}$. Since $\tau_i(a_i)=\tau_i(a_i-1)+\sigma_i(a_i)$ and $\tau_j(a_j)=\tau_j(a_j-\ 2)+\sigma_i(a_j-1)+\sigma_j(a_j)=\tau_j(a_j-2)+N_j$, this shows that

$$\eta_i\xi_j = q_{ji}^{N_j}\xi_j\eta_i.$$

Note: When $i=j$, similar calculations show that $\eta_i\xi_i=\xi_i\eta_i$. In this case we will have $\Phi(a_1,\cdots,a_i-3,\cdots,a_n)$ and observe that $\sigma_i(a_i)=\sigma_i(a_i-2)$.

Finally, consider,

$$
\begin{aligned}
&\eta_i\eta_j(\Phi(a_1,\cdots,a_n)) \\
&\quad = \quad \eta_i\left(\prod_{j<l} q_{lj}^{(\sigma_j(a_j)-1)\tau_l(a_l)}\prod_{l<j}(-1)^{a_l}q_{jl}^{\tau_l(a_l)}x_j^{\sigma_j(a_j)-1}\Phi(a_1,\cdots,a_j-1,\cdots,a_n)\right) \\
&\quad = \quad \prod_{j<l} q_{lj}^{(\sigma_j(a_j)-1)\tau_l(a_l)}\prod_{l<j}(-1)^{a_l}q_{jl}^{\tau_l(a_l)}x_j^{\sigma_j(a_j)-1}\prod_{i<m} q_{mi}^{(\sigma_i(a_i)-1)\tau_m(a_m)} \\
&\qquad\qquad \prod_{m<i}(-1)^{a_m}q_{im}^{\tau_m(a_m)}x_i^{\sigma_i(a_i)-1}\Phi(a_1,\cdots,a_i-1,\cdots,a_j-1,\cdots,a_n)
\end{aligned}
$$

$$
\begin{aligned}
&\eta_j\eta_i(\Phi(a_1,\cdots,a_n)) \\
&\quad = \quad \eta_j\left(\prod_{i<m} q_{mi}^{(\sigma_i(a_i)-1)\tau_m(a_m)}\prod_{m<i}(-1)^{a_m}q_{im}^{\tau_m(a_m)}x_i^{\sigma_i(a_i)-1}\Phi(a_1,\cdots,a_j-1,\cdots,a_n)\right)
\end{aligned}
$$

$$= \prod_{i<m} q_{mi}^{(\sigma_i(a_i)-1)\tau_m(a_m)} \prod_{m<i}(-1)^{a_m} q_{im}^{\tau_m(a_m)} x_i^{\sigma_i(a_i)-1} \prod_{j<l} q_{lj}^{(\sigma_j(a_j)-1)\tau_l(a_l)}$$
$$\prod_{l<j}(-1)^{a_l} q_{jl}^{\tau_l(a_l)} x_j^{\sigma_j(a_j)-1} \Phi(a_1, \cdots, a_i - 1, \cdots, a_j - 1, \cdots, a_n)$$

Comparison shows that a scalar factor for the term in which $l = i$ changes from $(-1)^{a_i} q_{ji}^{\tau_i(a_i)}$ to $(-1)^{a_i - 1} q_{ji}^{\tau_i(a_i-1)}$, and a scalar factor for the term in which $m = j$ changes from $q_{ji}^{(\sigma_i(a_i)-1)\tau_j(a_j)}$ to $q_{ji}^{(\sigma_i(a_i)-1)\tau_j(a_j-1)}$, and $x_j^{\sigma_j(a_j)-1} x_i^{\sigma_i(a_i)-1}$ is replaced by $x_j^{\sigma_j(a_j)-1} x_i^{\sigma_i(a_i)-1} = q_{ji}^{(\sigma_i(a_i)-1)(\sigma_j(a_j)-1)} x_i^{\sigma_i(a_i)-1} x_j^{\sigma_j(a_j)-1}$. Since $\tau_i(a_i) = \tau_i(a_i - 1) + \sigma_i(a_i)$ and $\tau_j(a_j) = \tau_j(a_j - 1) + \sigma_j(a_j)$, this shows that

$$\eta_i \eta_j = -q_{ji} \eta_j \eta_i.$$

Note: When $i = j$, we have $\eta_i^2(\Phi(a_1, \cdots, a_n)) = \rho x_i^{N_i - 2} \Phi(a_1, \cdots, a_i - 2, \cdots, a_n)$ where ρ is a nonzero scalar. If $N_i = 2$, then η_i^2 is a nonzero scalar multiple of ξ_i and the corresponding element in cohomology is zero if $N_i \neq 2$.

Thus any element in the algebra generated by the ξ_i and η_i may be written as a linear combination of elements of the form $\xi_1^{b_1} \cdots \xi_\theta^{b_\theta} \eta_1^{c_1} \cdots \eta_\theta^{c_\theta} \cdots \eta_n^{c_n}$ with $b_i \geq 0$ and $c_i \in \{0, 1\}$.

We claim that the set of all $\xi_1^{b_1} \cdots \xi_\theta^{b_\theta} \eta_1^{c_1} \cdots \eta_\theta^{c_\theta} \cdots \eta_n^{c_n}$ forms a k-basis for $H^*(S, k)$. So consider,

$$\begin{aligned}
&\xi_1^{b_1} \cdots \xi_\theta^{b_\theta} \eta_1^{c_1} \cdots \eta_\theta^{c_\theta} \cdots \eta_n^{c_n} (\Phi(2b_1 + c_1, \cdots, 2b_\theta + c_\theta, c_{\theta+1}, \cdots, c_n)) \\
&\quad = \xi_1^{b_1} \cdots \xi_\theta^{b_\theta} \eta_1^{c_1} \cdots \eta_\theta^{c_\theta} \cdots \eta_n^{c_n - 1} \Bigg(\prod_{l<n} q_{ln}^{(\sigma_n(c_n)-1)\tau_l(c_l)} x_n^{\sigma_n(c_n)-1} \\
&\qquad\qquad \cdot \Phi(2b_1 + c_1, \cdots, 2b_\theta + c_\theta, c_{\theta+1}, \cdots, c_n - 1) \Bigg)
\end{aligned}$$

If $c_n = 0$ there will be no term of $\eta_n^{c_n}$. In this case, we will get the same expression as above with n replaced by $n - 1$. If $c_n = 1$ then $c_n - 1 = 0$ and $\sigma_n(c_n) - 1 = 0$. Hence, we will get

$$\xi_1^{b_1}\cdots\xi_\theta^{b_\theta}\eta_1^{c_1}\cdots\eta_\theta^{c_\theta}\cdots\eta_n^{c_n}(\Phi(2b_1+c_1,\cdots,2b_\theta+c_\theta,c_{\theta+1},\cdots,c_{n-1},c_n))$$
$$=\xi_1^{b_1}\cdots\xi_\theta^{b_\theta}\eta_1^{c_1}\cdots\eta_\theta^{c_\theta}\cdots\eta_{n-1}^{c_{n-1}}(\mu\Phi(2b_1+c_1,\cdots,2b_\theta+c_\theta,c_{\theta+1},\cdots,c_{n-1},0))$$

where μ is some nonzero scalar.

Similarly, after applying all the η_i, we will finally get

$$\xi_1^{b_1}\cdots\xi_\theta^{b_\theta}\eta_1^{c_1}\cdots\eta_\theta^{c_\theta}\cdots\eta_n^{c_n}(\Phi(2b_1+c_1,\cdots,2b_\theta+c_\theta,c_{\theta+1},\cdots,c_n))$$
$$=\xi_1^{b_1}\cdots\xi_\theta^{b_\theta}(\nu_1\Phi(2b_1,\cdots,2b_\theta,0,\cdots,0,0))$$

where ν_1 is some nonzero scalar. Next we apply ξ_θ and get

$$\xi_1^{b_1}\cdots\xi_\theta^{b_\theta}\eta_1^{c_1}\cdots\eta_\theta^{c_\theta}\cdots\eta_n^{c_n}(\Phi(2b_1+c_1,\cdots,2b_\theta+c_\theta,c_{\theta+1},\cdots,c_n))$$
$$=\xi_1^{b_1}\cdots\xi_\theta^{b_\theta-1}(\nu_1\prod_{m>\theta}q_{\theta l}^{N_\theta\tau_m(0)}\Phi(2b_1,\cdots,2b_{\theta-1},2b_\theta-2,0,\cdots,0,0))$$

If we keep applying ξ_θ again and again we will get

$$\xi_1^{b_1}\cdots\xi_\theta^{b_\theta}\eta_1^{c_1}\cdots\eta_\theta^{c_\theta}\cdots\eta_n^{c_n}(\Phi(2b_1+c_1,\cdots,2b_\theta+c_\theta,c_{\theta+1},\cdots,c_n))$$
$$=\xi_1^{b_1}\cdots\xi_{\theta-1}^{b_{\theta-1}}(\nu_2\Phi(2b_1,\cdots,2b_{\theta-1},2b_\theta-2b_\theta,0,\cdots,0,0))$$
$$=\xi_1^{b_1}\cdots\xi_{\theta-1}^{b_{\theta-1}}(\nu_2\Phi(2b_1,\cdots,2b_{\theta-1},0,0,\cdots,0,0))$$

where ν_2 is some nonzero scalar. Finally after applying all the other ξ_i we get

$$\xi_1^{b_1}\cdots\xi_\theta^{b_\theta}\eta_1^{c_1}\cdots\eta_\theta^{c_\theta}\cdots\eta_n^{c_n}(\Phi(2b_1+c_1,\cdots,2b_\theta+c_\theta,c_{\theta+1},\cdots,c_n))$$
$$=\nu\Phi(0,\cdots,0)$$

where ν is some nonzero scalar.

Next consider $\xi_1^{b_1}\cdots\xi_\theta^{b_\theta}\eta_1^{c_1}\cdots\eta_\theta^{c_\theta}\cdots\eta_n^{c_n}(\Phi(e_1,\cdots,e_\theta,e_{\theta+1},\cdots,e_n))$ where $e_i\neq$

$2b_i + c_i$ for some i. Then

$$\xi_1^{b_1} \cdots \xi_\theta^{b_\theta} \eta_1^{c_1} \cdots \eta_\theta^{c_\theta} \cdots \eta_n^{c_n} (\Phi(e_1, \cdots, e_\theta, e_{\theta+1}, \cdots, e_n)) = 0$$

To see this, let us consider an example. Let $n = 7,\ \theta = 3,\ b_1 = 2,\ b_2 = 0,\ c_1 = c_2 = c_3 = 1$. Then

$$\begin{aligned} \xi_1^2 \eta_1 \eta_2 \eta_3 (\Phi(5,1,1)) &= \nu_1 \xi_1^2 \eta_1 \eta_2 (\Phi(5,1,0) \\ &= \nu_2 \xi_1^2 \eta_1 (\Phi(5,0,0) \\ &= \cdots = \nu \Phi(0,0,0) \end{aligned}$$

On the other hand, a given partition $(3,3,1)$ also adds up to degree 7 and $\xi_1^2 \eta_1 \eta_2 \eta_3 (\Phi(3,3,1)) = 0$.

That is, $\xi_1^{b_1} \cdots \xi_\theta^{b_\theta} \eta_1^{c_1} \cdots \eta_\theta^{c_\theta} \cdots \eta_n^{c_n}$ takes all other S-basis elements of $K_{\sum(2b_i+c_i)}$ to 0. Therefore, all such monomials form a linearly independent set.

Finally to show that it forms a basis we will show one to one correspondence between the tuples $(b_1, \cdots, b_\theta, c_1, \cdots, c_n)$ and $(a_1, \cdots, a_n)$ where we know that for each n-tuple $(a_1, \cdots, a_n)$ of non-negative integers with $a_i = 0$ or 1 for each i, $\theta + 1 \le i \le n$, $\Phi(a_1, \cdots, a_n)$ is a free generator in degree $a_1 + \cdots + a_n$.

Consider $(a_1, \cdots, a_\theta, \cdots, a_n)$. If a_i is odd, then $a_i = 2b_i + c_i$ for some integer b_i and $c_i = 1$. If a_i is even, then $a_i = 2b_i$ for some integer b_i and $c_i = 0$.

On the other hand consider $(2b_1 + c_1, \cdots, 2b_\theta + c_\theta, c_{\theta+1}, \cdots, c_n)$. We know that $c_i = 0$ or $c_i = 1$ and b_i is some integer. So choose $a_i = 2b_i + c_i$.

Hence, we get that $H^*(S,k) \cong Hom_S(K_\bullet, k) \cong Hom_k(V,k)$ where V has basis all $\Phi(a_1, \cdots, a_n)$. This shows that the set of monomials of the form $\xi_1^{b_1} \cdots \xi_\theta^{b_\theta} \cdots \xi_n^{b_n} \eta_1^{c_1} \cdots \eta_\theta^{c_\theta} \cdots \eta_n^{c_n}$ forms a k-basis for $H^*(S,k)$.

$\square$

CHAPTER IV

SOME COCYCLES ON THE ALGEBRA

For this chapter we will use the same terminology as used by Mastnak and Witherspoon in Section 6 of [16] with some additional information.

Let B be a PBW algebra over k as defined in Chapter 2 and $A = B/(x_1^{N_1}, \cdots, x_\theta^{N_\theta})$. As a vector space B has a basis $\{x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n} \mid i_1, \cdots, i_n \in \mathbb{N}\}$.

Let $b \in B$. Then $b = \sum_I a_I x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ is a finite sum where $I = (i_1, i_2, \cdots, i_n)$ and a_I is a scalar. Therefore,

$$\begin{aligned} b + (x_1^{N_1}, \cdots, x_\theta^{N_\theta}) &= \sum_I a_I x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n} + (x_1^{N_1}, \cdots, x_\theta^{N_\theta}) \\ &= \sum_{\substack{I \\ 0\le i_j<N_j \\ 1\le j\le\theta}} (a_I x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n} + (x_1^{N_1}, \cdots, x_\theta^{N_\theta})) \end{aligned}$$

This proves that $\{x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n} \mid 0 \le i_1 < N_1, \cdots, 0 \le i_\theta < N_\theta,\ i_{\theta+1}, \cdots, i_n \in\ \mathbb{N}\}$ is a spanning set for A.

Define,

$$[x_1^{i_1}\cdots x_n^{i_n}, x_1^{j_1}\cdots x_n^{j_n}]_c = x_1^{i_1}\cdots x_n^{i_n}x_1^{j_1}\cdots x_n^{j_n} - (\prod_{k<l} q_{lk}^{-(j_l i_k - j_k i_l)})x_1^{j_1}\cdots x_n^{j_n}x_1^{i_1}\cdots x_n^{i_n}.$$

Definition IV.1. *An element of the form $x_1^{i_1}\cdots x_n^{i_n}$ is said to be in the "braided" center of B, if*

$$[x_1^{i_1}\cdots x_n^{i_n}, x_1^{j_1}\cdots x_n^{j_n}]_c = 0, \ \textit{for all } x_1^{j_1}\cdots x_n^{j_n} \in\ B. \tag{IV.1}$$

To prove linear independence we need to assume that $x_i^{N_i}$ is in the braided center of B for all $i,\ 1 \le i \le \theta$. This assumption will be also needed for a later part of this chapter.

To show that the set is linearly independent we need to prove that $\sum_I a_I x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ belonging to $(x_1^{N_1}, \cdots, x_\theta^{N_\theta})$ implies all $a_I = 0$.

Consider,

$$\sum_I a_I x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} = \sum_{J,i} T_J x_i^{N_i} W_J$$

where $T_J, W_J \in B$. Since $x_i^{N_i}$ is in the braided center we have

$$\sum_I a_I x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} = \sum_{J,i} x_i^{N_i} U_J$$

where $U_J \in B$. Observe that in each expression on the right hand side there is atleast one i for which the power of x_i is atleast N_i. Thus by comparing the coefficients we get $a_I = 0$.

Hence, $\{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \mid 0 \leq i_1 < N_1, \cdots, 0 \leq i_\theta < N_\theta,\ i_{\theta+1}, \cdots, i_n \in \mathbb{N}\}$ is a basis for A.

Next we want to define 2-cocycles ξ_i on A. These 2-cocycles represent the elements of $H^2(A, k)$. We make use of the reduced bar resolution of k,

$$\cdots \longrightarrow B \otimes (B^+)^{\otimes 2} \xrightarrow{\delta_2} B \otimes B^+ \xrightarrow{\delta_1} B \xrightarrow{\varepsilon} k \longrightarrow 0.$$

where B is an augmented algebra with augmention map $\varepsilon : B \to k$, $B^+ = \operatorname{Ker} \varepsilon$ is the augmentation ideal and $\delta_i(b_0 \otimes b_1 \otimes \cdots \otimes b_i) = \sum_{j=0}^{i-1} (-1)^j b_0 \otimes \cdots \otimes b_j b_{j+1} \otimes \cdots \otimes b_i$. For each $i, 1 \leq i \leq \theta$ define $\tilde{\xi}_i : B^+ \otimes B^+ \to k$ by

$$\tilde{\xi}_i(r \otimes s) = \gamma_{(0,\cdots,0,N_i,0,\cdots,0)}$$

where N_i is in the i^{th} position and $rs = \sum_a \gamma_a x^a \in B$. We need to check that $\tilde{\xi}_i(r \otimes s)$ is associative that is to show that $\tilde{\xi}_i(rr_1 \otimes s) = \tilde{\xi}_i(r \otimes r_1 s)$ for all $r, r_1, s \in B^+$. But this is true by definition and thus $\tilde{\xi}_i$ may be trivially extended to a 2-cocycle on B. Let us see how it is done. We will denote the 2-cocycle on B by $\tilde{\xi}_i$ and define as

$\tilde{\xi}_i(b_1 \otimes b_2) = \tilde{\xi}_i \mid_{B^+ \otimes B^+} (b_1 \otimes b_2)$ for $b_1, b_2 \in B^+$. Indeed $\tilde{\xi}_i$ is a coboundary on B that is $\tilde{\xi}_i = -\delta^* h_i$ where $h_i(r)$ when written as a linear combination of PBW basis elements is the coefficient of $x_i^{N_i}$ in $r \in B^+$. To see this note that $h_i : B \otimes B^+ \to k$ is a 1-cochain, $Hom_B(B \otimes B^+, k) \cong Hom_k(B^+, k)$ and $\delta^* h_i \in Hom_B(B \otimes B^+ \otimes B^+, k)$.

$$
\begin{aligned}
\delta^* h_i(r \otimes s) &= \delta^* h_i(1 \otimes r \otimes s) \\
&= h_i(\delta(1 \otimes r \otimes s)) \\
&= h_i(r \otimes s - 1 \otimes rs) \\
&= r h_i(s) - h_i(rs) \\
&= \varepsilon(r) h_i(s) - h_i(rs) \qquad (\text{ since } r \text{ acts as multiplication by } \varepsilon(r))
\end{aligned}
$$

since $h_i(s) \in k$ and $\varepsilon(r) = 0$ we have $\delta^* h_i(r \otimes s) = 0 - h_i(rs) = -h_i(rs)$.

To define a 2-cocycle ξ_i on A we next show that $\tilde{\xi}_i$ factors through the quotient map $\pi : B \to A$ and that ξ_i is not a coboundary on A. We must show that $\tilde{\xi}_i(r, s) = 0$ whenever either r or $s \in \text{Ker } \pi$. Consider the following diagram

$$
\begin{tikzcd}
B^+ \otimes B^+ \arrow[r, "\tilde{\xi}_i"] \arrow[d, "\pi \otimes \pi"'] & k \\
A \otimes A \arrow[ur, dotted, "\xi_i"'] &
\end{tikzcd}
$$

Suppose $x^a \in \text{Ker } \pi$ then $a_j \geq N_j$ for some j with $1 \leq j \leq \theta$. As per the assumption that $x_i^{N_i}$ is in the braided center we can write $x^a = \vartheta x_j^{N_j} x^b$ where ϑ is a non-zero scalar and b may be 0. Therefore, $\tilde{\xi}_i(x^a \otimes x^c) = \vartheta \tilde{\xi}_i(x_j^{N_j} x^b \otimes x^c)$ and this is the coefficient of $x_i^{N_i}$ in the product $\vartheta x_j^{N_j} x^b x^c$. If $j = i$, then since $x^c = x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n} \in B^+$ the above product cannot have non-zero coefficient for $x_i^{N_i}$. The same is true, if $j \neq i$ since $x_j^{N_j}$ is a factor of $x^a x^c$. If $x^c \in \text{Ker } \pi$ a similar argument will work.

Thus, we have $\tilde{\xi}_i(x^a \otimes x^c) = 0$ that is, $\tilde{\xi}_i$ factors through the quotient map $\pi : B \to A$.

Therefore, we may define $\xi_i : A^+ \otimes A^+ \to k$ by

$$\xi_i(r \otimes s) = \tilde{\xi}_i(\tilde{r} \otimes \tilde{s})$$

where $\tilde{r}, \tilde{s}$ are defined via a section of π. (Choose the section ϕ of the quotient map $\pi : B \to A$ such that $\phi(r) = \tilde{r}$ where $\tilde{r}$ is the unique element that is a linear combination of the PBW basis elements of B with $i_l < N_l$ for all $l = 1, \cdots, n$).

This is well defined since $\tilde{\xi}_i$ is well defined. We still need to verify that ξ_i is associative on A^+. Let $r, s, u \in A^+$ and since π is algebra homomorphism $\tilde{r}\tilde{s} = \widetilde{rs} + y$ and $\tilde{s}\tilde{u} = \widetilde{su} + z$ for some $y, z \in$ Ker π. Observe that Ker $\pi \otimes B + B \otimes$ Ker $\pi \subset$ Ker $\tilde{\xi}_i$. Therefore, we have

$$\begin{aligned}
\xi_i(rs \otimes u) &= \tilde{\xi}_i(\widetilde{rs} \otimes \tilde{u}) \\
&= \tilde{\xi}_i((\tilde{r}\tilde{s} - y) \otimes \tilde{u}) \\
&= \tilde{\xi}_i(\tilde{r}\tilde{s} \otimes \tilde{u}) \\
&= \tilde{\xi}_i(\tilde{r} \otimes \tilde{s}\tilde{u}) \quad (\tilde{\xi}_i \text{ associative }) \\
&= \tilde{\xi}_i(\tilde{r} \otimes \widetilde{su}) \\
&= \xi_i(r \otimes su)
\end{aligned}$$

This shows that ξ_i is associative on A^+. Hence, ξ_i is 2-cocycle on A.

CHAPTER V

FINITE GENERATION

In this chapter we prove our main theorem. We follow the same terminology as used in Section 5 of [15] with some additional information.

Let B be a PBW algebra as defined in Chapter 2 and $A = B/(x_1^{N_1}, \cdots, x_\theta^{N_\theta})$. Recall the assumption from Chapter 4 that $x_i^{N_i}$ is in the braided center. Hence, a filtration on B induces a filtration on A [4, Theorem 4.6.5] for which $S = GrA$, given by generators and relations of type (III.1). Thus $H^*(S,k)$ is given by Theorem III.1.

Now our algebra A is an augmented algebra over a field k, with augmentation $\varepsilon : A \to k$. Since A is filtered it induces an increasing filtration $F_0P_\bullet \subset F_1P_\bullet \subset \cdots \subset F_nP_\bullet \subset \cdots$ on the reduced bar (free A) resolution of k,

$$P_\bullet : \cdots \xrightarrow{\partial_3} A \otimes (A^+)^{\otimes 2} \xrightarrow{\partial_2} A \otimes A^+ \xrightarrow{\partial_1} A \xrightarrow{\varepsilon} k \to 0$$

where $A^+ = \text{Ker } \varepsilon$, $\partial_n(a_0 \otimes \cdots \otimes a_n) = \sum_{j=0}^{n-1}(-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n$ and the filtration is given in each degree n by

$$F_p(A \otimes (A^+)^{\otimes n}) = \sum_{i_0+\cdots+i_n=p} F_{i_0}A \otimes F_{i_1}(A^+) \otimes \cdots \otimes F_{i_n}(A^+).$$

Then the reduced bar complex of GrA is precisely $GrP_\bullet$, where

$$(GrP_n)_p := F_pP_n/F_{p-1}P_n.$$

Now let $\mathcal{C}^\bullet(A) := Hom_A(P_\bullet, k)$. Note that $\mathcal{C}^n(A) = Hom_A(P_n, k) = Hom_A(A \otimes (A^+)^{\otimes n}, k)$ is a filtered vector space where

$$F^p\mathcal{C}^n(A) = \{f : P_n \to k \mid f\,|_{F_{p-1}P_n} = 0\}$$

This filtration is compatible with the coboundary map on $\mathcal{C}^\bullet(A)$. Hence, $\mathcal{C}^\bullet(A)$ is a filtered cochain complex: $\mathcal{C} = F^0\mathcal{C}^\bullet \supset F^1\mathcal{C}^\bullet \supset \cdots$. Now our algebra A satisfies $F_pA = 0$ if $p < 0$, $1 \in F_0A$ and $A = \bigcup_p F_pA$. Thus, there is a convergent spectral sequence associated to the filtration of a cochain complex (see [17, Theorem 3] and [18, Theorem 12.5]):

$$E_1^{p,q} = H^{p+q}((GrA)_p, k) \Longrightarrow H^{p+q}(A, k). \tag{V.1}$$

Note: For special cases refer to [20, Theorem 5.5.1].

From Chapter 4 we know that

$$\xi_i(x^a \otimes x^b) = \gamma_i \tag{V.2}$$

where γ_i is the coefficient of $x_i^{N_i}$ in the product $x^a x^b$, and x^a, x^b range over all pairs of PBW basis elements. Also, observe that ξ_i are in degrees $(p_i, 2-p_i)$ where p_i denotes the total order on PBW basis element.

We wanted to relate these functions ξ_i to the elements of the E_1 page of the spectral sequence (V.1). We have $\xi_i \mid_{F_{p_i-1}(A\otimes A)} = 0$ but $\xi_i \mid_{F_{p_i}(A\otimes A)} \neq 0$ by (V.2).

Thus, we conclude by the definition of ξ_i from Chapter 4 that $\xi_i \in F^{p_i}\mathcal{C}^2$ but $\xi_i \notin F^{p_i+1}\mathcal{C}^2$. The filtration on $\mathcal{C}^\bullet$ induces a filtration on $H^*(\mathcal{C}^\bullet)$, that is to say $F^pH^n(\mathcal{C}^\bullet) := \mathrm{im}\{H^n(F^p\mathcal{C}^\bullet) \to H^n(\mathcal{C}^\bullet)\}$ with $F^0H^n(\mathcal{C}^\bullet) = H^n(\mathcal{C}^\bullet)$. By denoting the corresponding cocycle in $F^{p_i}H^2(A, k)$ by the same letter we further conclude that $\xi_i \in \mathrm{im}\{H^2(F^{p_i}\mathcal{C}^\bullet) \to H^2(\mathcal{C}^\bullet)\} = F^{p_i}H^2(A, k)$, but $\xi_i \notin \mathrm{im}\{H^2(F^{p_i+1}\mathcal{C}^\bullet) \to H^2(\mathcal{C}^\bullet)\} = F^{p_i+1}H^2(A, k)$. Hence, we can identify ξ_i with corresponding nontrivial homogeneous element in the associated graded complex:

$$\tilde{\xi}_i \in F^{p_i}H^2(A, k)/F^{p_i+1}H^2(A, k) \simeq E_\infty^{p_i, 2-p_i}.$$

Refer to [17] for the isomorphism.

Since $\xi_i \in F^{p_i}\mathcal{C}^2$ but $\xi_i \notin F^{p_i+1}\mathcal{C}^2$, it induces an element $\bar{\xi}_i \in E_0^{p_i,2-p_i} = F^{p_i}\mathcal{C}^2/F^{p_i+1}\mathcal{C}^2$ which will be in the kernels of all the differentials of the spectral sequence since it is induced by an actual cocycle in $\mathcal{C}^\bullet$. Hence, the image of $\bar{\xi}_i$ will be in the E_∞-page. Now the non-zero element $\tilde{\xi}_i$ is also induced by the same cocycle as $\bar{\xi}_i$ in $\mathcal{C}^\bullet$. Hence we may identify these cocycles. This leads to the conclusion that $\tilde{\xi}_i \in E_0^{p_i,2-p_i}$, and, correspondingly, its image in $E_1^{p_i,2-p_i} \hookrightarrow H^2(GrA,k)$ which we denote by the same symbol, is a permanent cycle.

Note that via the formula (V.2) we can obtain similar cocycles $\hat{\xi}_i$ for $S = GrA$. Comparing the values of $\bar{\xi}_i$ and $\hat{\xi}_i$ on basis elements $x^a \otimes x^b$ of $GrA \otimes GrA$ leads us to the conclusion that they are the same function. Hence $\hat{\xi}_i \in E_1^{p_i,2-p_i}$ are permanent cycles.

We will identify these elements $\hat{\xi}_i \in H^2(GrA,k)$ with the cohomology classes $\xi_i \in H^*(S,k)$ of Theorem III.1.

Theorem V.1. *For each i $(1 \leq i \leq n)$, the cohomology classes ξ_i and $\hat{\xi}_i$ coincide as elements of $H^2(GrA,k)$.*

Proof. In Chapter 3 we have defined the chain complex $K_\bullet$ which is a projective resolution of the trivial GrA-module k. Elements $\eta_i \in H^1(GrA,k)$ and $\xi_i \in H^2(GrA,k)$ were defined via the complex $K_\bullet$. Our aim is to identify ξ_i with the elements of the chain complex $\mathcal{C}^\bullet$ defined above. For this we consider the following diagram and define the maps F_1, F_2 making it commutative, where $S = GrA$:

$$\begin{tikzcd}
\cdots \arrow[r] & K_2 \arrow[r, "d"] \arrow[d, "F_2"] & K_1 \arrow[r, "d"] \arrow[d, "F_1"] & K_0 \arrow[r, "\varepsilon"] \arrow[d, equal] & k \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & S \otimes (S^+)^{\otimes 2} \arrow[r, "\partial_2"] & S \otimes (S^+) \arrow[r, "\partial_1"] & S \arrow[r, "\varepsilon"] & k \arrow[r] & 0
\end{tikzcd}$$

where the map $d = d_1 + d_2 + \cdots d_n$ is defined in Chapter 3 and $\partial_i(s_0 \otimes s_1 \otimes \cdots \otimes s_i) = \sum_{j=0}^{i-1}(-1)^j s_0 \otimes \cdots \otimes s_j s_{j+1} \otimes \cdots \otimes s_i$ is defined in Chapter 4. Let $\Phi(\cdots 1_i \cdots)$ where

1 is in the ith position and 0 in all other positions denote the basis element of K_1, $\Phi(\cdots 1_i \cdots 1_j \cdots)$ (respectively $\Phi(\cdots 2_i \cdots)$ for $i \leq \theta$) where 1 is in the ith and jth positions ($i \neq j$), and 0 in all other positions (respectively a 2 in the ith position and 0 in all other positions) denote the basis element of K_2. Let

$$\begin{aligned} F_1(\Phi(\cdots 1_i \cdots)) &= 1 \otimes x_i, \\ F_2(\Phi(\cdots 2_i \cdots)) &= \sum_{a_i=0}^{N_i-2} x_i^{a_i} \otimes x_i \otimes x_i^{N_i-a_i-1}, \\ F_2(\Phi(\cdots 1_i \cdots 1_j \cdots)) &= 1 \otimes x_j \otimes x_i - q_{ji} \otimes x_i \otimes x_j \end{aligned}$$

We want to provide a chain map $F_\bullet : K_\bullet \to S \otimes (S^+)^{\otimes \bullet}$ by extending F_1, F_2 to maps $F_i : K_i \to S \otimes (S^+)^{\otimes i}, i \geq 1$. This can be done by showing that the two nontrivial squares in the above diagram commute.

Consider,

$$\begin{aligned} d(\Phi(\cdots 1_i \cdots)) &= (d_1 + \cdots + d_i + \cdots + d_n)(\Phi(\cdots 1_i \cdots)) \\ &= x_i \Phi(\cdots 0_i \cdots) \\ &= x_i \end{aligned}$$

$$\begin{aligned} \partial_1 \circ F_1(\Phi(\cdots 1_i \cdots)) &= \partial_i(1 \otimes x_i) \\ &= 1 \cdot x_i \\ &= x_i \end{aligned}$$

Thus, we have $d = \partial_1 \circ F_1$.

Next consider,

$$
\begin{aligned}
F_1 \circ d(\Phi(\cdots 1_i \cdots 1_j \cdots)) &= F_1(-q_{ji}x_i(\cdots 0_i \cdots 1_j \cdots) + x_j(\cdots 1_i \cdots 0_j \cdots)) \\
&= -q_{ji}x_i \otimes x_j + x_j \otimes x_i
\end{aligned}
$$

$$
\begin{aligned}
\partial_2 \circ F_2(\Phi(\cdots 1_i \cdots 1_j \cdots)) &= \partial_2(1 \otimes x_j \otimes x_i - q_{ji} \otimes x_i \otimes x_j) \\
&= 1 \cdot x_j \otimes x_i - 1 \otimes x_j x_i - q_{ji}x_i \otimes x_j + q_{ji} \otimes x_i x_j \\
&= x_j \otimes x_i - 1 \otimes q_{ji}x_i x_j - q_{ji}x_i \otimes x_j + q_{ji} \otimes x_i x_j \\
&= x_j \otimes x_i - q_{ji} \otimes x_j x_i - q_{ji}x_i \otimes x_j + q_{ji} \otimes x_j x_i \\
&= x_j \otimes x_i - q_{ji}x_i \otimes x_j
\end{aligned}
$$

Finally, consider

$$
\begin{aligned}
F_1 \circ d(\Phi(\cdots 2_i \cdots)) &= F_1(x_i^{N_i-1}\Phi(\cdots 1_i \cdots)) \\
&= x_i^{N_i-1} \otimes x_i
\end{aligned}
$$

$$
\begin{aligned}
\partial_2 \circ F_2(\Phi(\cdots 2_i \cdots)) &= \partial_2(\sum_{a_i=0}^{N_i-2} x_i^{a_i} \otimes x_i \otimes x_i^{N_i-a_i-1}) \\
&= \partial_2(1 \otimes x_i \otimes x_i^{N_i-1} + x_i \otimes x_i \otimes x_i^{N_i-2} + \cdots + x_i^{N_i-2} \otimes x_i \otimes x_i) \\
&= (x_i \otimes x_i^{N_i-1} - 1 \otimes x_i^{N_i}) + (x_i^2 \otimes x_i^{N_i-2} - x_i \otimes x_i^{N_i-1}) + \cdots + \\
&\qquad (x_i^{N_i-2} \otimes x_i^2 - x_i^{N_i-3} \otimes x_i^3) + (x_i^{N_i-1} \otimes x_i - x_i^{N_i-2} \otimes x_i^2) \\
&= x_i^{N_i-1} \otimes x_i
\end{aligned}
$$

Thus, we have $F_1 \circ d = \partial_2 \circ F_2$.

Hence, two nontrivial squares in the above diagram commute. So by the Comparison Theorem [13] there exists a chain map $F_\bullet : K_\bullet \to S \otimes (S^+)^{\otimes \bullet}$ that induces an isomorphism on cohomology.

We now verify that the maps F_1, F_2 give the desired identifications. Here we use the definition in V.2 to represent the function ξ_i on the reduced bar complex, $\xi_i(1 \otimes x^a \otimes x^b) := \xi_i(x^a \otimes x^b)$. Then

$$\begin{aligned}
F_2^*(\xi_i)(\Phi(\cdots 2_i \cdots)) &= \xi_i(F_2(\Phi(\cdots 2_i \cdots))) \\
&= \xi_i(\sum_{a_i=0}^{N_i-2} x_i^{a_i} \otimes x_i \otimes x_i^{N_i-a_i-1}) \\
&= \sum_{a_i=0}^{N_i-2} \varepsilon(x_i^{a_i}) \xi_i(1 \otimes x_i \otimes x_i^{N_i-a_i-1}) \\
&= \xi_i(x_i \otimes x_i^{N_i-1}) \\
&= 1
\end{aligned}$$

Further, we check that $F_2^*(\xi_i)(\Phi(\cdots 1_i \cdots 1_j \cdots)) = 0$ for all i, j and $F_2^*(\xi_i)(\Phi(\cdots 2_j \cdots)) = 0$ for all $j \neq i$.

Consider,

$$\begin{aligned}
1)\ F_2^*(\xi_i)(\Phi(\cdots 1_i \cdots 1_j \cdots)) &= \xi_i(F_2(\Phi(\cdots 1_i \cdots 1_j \cdots))) \\
&= \xi_i(1 \otimes x_j \otimes x_i - q_{ji} \otimes x_i \otimes x_j) \\
&= \xi_i(x_j \otimes x_i - q_{ji} x_i \otimes x_j) \\
&= \xi_i(x_j \otimes x_i) - q_{ji} \xi_i(x_i \otimes x_j) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
2)\ F_2^*(\xi_i)(\Phi(\cdots 2_j \cdots)) &= \xi_i(F_2(\Phi(\cdots 2_j \cdots))) \\
&= \xi_i(\sum_{a_j=0}^{N_j-2} x_i^{a_j} \otimes x_j \otimes x_j^{N_j-a_j-1}) \\
&= \sum_{a_j=0}^{N_j-2} \varepsilon(x_j^{a_j})\xi_i(1 \otimes x_j \otimes x_j^{N_j-a_j-1}) \\
&= \xi_i(x_j \otimes x_j^{N_j-1}) \\
&= 0
\end{aligned}
$$

Therefore $F_2^*(\xi_i)$ is the dual function to $\Phi(\cdots 2_i \cdots)$ which is precisely ξ_i.

$\square$

In the same manner, we identify the elements η_i defined above with functions at the chain level in cohomology. For that define

$$
\eta_i(x^a) = \begin{cases} 1, & \text{if } x^a = x_i \\ 0, & \text{otherwise} \end{cases}
$$

The functions η_i represent a basis of $H^1(S,k) \simeq Hom_k(S^+/(S^+)^2, k)$. Consider,

$$
\begin{aligned}
F_1^*(\eta_i)(\Phi(\cdots 1_j \cdots)) &= \eta_i(F_1(\Phi(\cdots 1_j \cdots))) \\
&= \eta_i(1 \otimes x_j) \\
&= \eta_i(x_j) \\
&= \begin{cases} 1, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases} \\
&= \delta_{ij}
\end{aligned}
$$

Thus $F_1^*(\eta_i)$ is the dual function to $\Phi(\cdots 1_i \cdots)$ which is precisely η_i. Therefore η_i and $\hat{\eta}_i$ coincide as elements of $H^1(S,k)$ where $\hat{\eta}_i$ is a 1-cocycle of A.

Theorem V.2. *The cohomology algebra $H^*(A, k)$ is finitely generated over the subalgebra generated by all ξ_i.*

Proof. Let $E_1^{*,*} \Longrightarrow H^*(A, k)$ be the May spectral sequence and $D^{*,*}$ be the bigraded subalgebra of $E_1^{*,*}$ generated by the elements ξ_i. So by the above discussion $D^{*,*}$ consists of permanent cycles and ξ_i is in bidegree $(p_i, 2 - p_i)$. Moreover, $D^{*,*}$ is Noetherian since it is a quantum polynomial algebra in ξ_i [10]. By Theorem III.1 the algebra $E_1^{*,*}$ is generated by ξ_i and η_i where the generators η_i are nilpotent. Since $D^{*,*}$ is a subalgebra of $E_1^{*,*}$, we get an inclusion map $f : D^{*,*} \to E_1^{*,*}$ making $E_1^{*,*}$ a module over $D^{*,*}$. Hence, $E_1^{*,*}$ is a finitely generated module over $D^{*,*}$ and is generated by $\eta_1, \cdots, \eta_n$. Therefore, by Lemma II.12, E_∞^* is a Noetherian $\mathrm{Tot}(D^{*,*})$-module. But $E_\infty^* \cong GrH^*(A, k)$ [18]. Thus, $GrH^*(A, k)$ is a Noetherian $\mathrm{Tot}(D^{*,*})$-module and hence is finitely generated. Therefore, $H^*(A, k)$ is finitely generated.

$\square$

CHAPTER VI

CONCLUSION

In this dissertation we used techniques of Mastnak, Pevtsova, Schauenburg and Witherspoon [15] to prove cohomology of A is finitely generated. Here, the algebra A is the quotient $B/(x_1^{N_1}, \cdots, x_\theta^{N_\theta})$ where B is a PBW algebra generated by $x_1, \cdots, x_\theta, \cdots, x_n$ and $N_i > 1,\ 1 \leq i \leq \theta$. Since we choose our parameters that are not necessarily roots of unity and allow non-nilpotent generators, our result is a generalization of Mastnak, Pevtsova, Schauenburg and Witherspoon's [15] result. Moreover, we deal with PBW algebras in general, whereas they looked at those that arise from subalgebras of pointed Hopf algebras.

First, in Chapter 3 we have shown that cohomology of a quotient of a quantum symmetric algebra S is finitely generated. This helped us to conclude that cohomology of the associated graded algebra GrA is finitely generated because the filtration on A, which is induced by the filtration on B, has the property that $GrA = S$. Finally, in Chapter 5 we proved our main theorem that is, $H^*(A, k)$ is finitely generated with the help of the finite generation lemma stated in Chapter 2 and a spectral sequence argument.

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